

Acknowledgement

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Contents

1	Definitions and Generals properties	5
1.1	Lebesgue Spaces	5
1.1.1	Definitions and Properties of $L^p(\Omega)$ Spaces	5
1.2	Sobolev Spaces	6
1.2.1	Generals Definitions	6
1.2.2	Sobolev Imbedding	7
1.3	Some Useful Inequalities	8
1.4	Elliptic Problems and Regularity	9
2	Singular Linear Dirichlet Problem	11
2.1	Regularity of Weak Solutions	11
2.2	Existence and Uniqueness of Solutions	16
2.3	Non-existence Result for $\beta \geq 2$	20
3	Existence and uniqueness of Singular Problem	22
3.1	Existence for The case $0 < \alpha < 1$	23
3.1.1	Ordered Sub and Supersolutions	31
3.2	Uniqueness of Solutions for $0 < \alpha < 1$	32
3.3	Existence for The case $\alpha = 1$	32
3.3.1	Ordered Sub and Supersolutions	39
3.4	Uniqueness for $\alpha = 1$	40
3.5	Existence for The case $1 < \alpha < 3$	40
3.5.1	Ordered Sub and Supersolutions	45

<i>CONTENTS</i>	2
3.6 Uniqueness for $1 < \alpha < 3$	46
3.7 Existence for The case $\alpha \geq 3$	47
3.7.1 Ordered of Sub and Supersolutions	53
3.8 Uniqueness for $\alpha \geq 3$	54
bibliography	55

Introduction

Nonlinear elliptic singular problem have been studied during the last forty years in what concerns existence, uniqueness (or multiplicity) and regularity of positive weak solutions. We will deal with the following problem :

$$\begin{cases} -\Delta u &= \frac{p(x)}{u^\alpha} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \end{cases} \quad (1)$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^n$, $\alpha > 0$ and p is a non negative function. The first relevant existence results for a class of problems including the model case (1) with p smooth and $\alpha > 0$, were obtained in two important papers by crandall -Rabinowitz-Tartar [5] and Stuart [13]. Concerning weak solutions in the usual Sobolev spaces, Laser -Mckenna [12] proved that the classical solution belongs to $H_0^1(\Omega)$ if and only if $0 \leq \alpha < 3$. This result was generalized later for $p(x) = d(x)^\beta$ with $d(x) = d(x, \partial\Omega)$ with the restrictions $\beta > -2$ by Zhang-Cheng [14].

In this work, we will study especially the problem (1) with $p(x) = 1$. For this, we divide our work to three chapters which are:

In the first chapter we recall some definitions and generals properties about Lebesgue and Sobolev spaces which will be useful in this work.

In the second chapter we deal with the linear elliptic problem,

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega \end{cases} \quad (P)$$

Where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^N$, $f : \Omega \rightarrow \mathbb{R}$ is continuous function, such that $0 \leq f(x) \leq d(x)^{-\beta}$. Where $d(x) = d(x, \partial\Omega)$. We

study the existence and the regularity of the weak solutions to (P) with $0 < \beta < 2$ and we prove the non existence of solutions for $\beta \geq 2$.

In the last chapter we study the existence and uniqueness of weak solution to the elliptic singular quasilinear problem,

$$\begin{cases} -\Delta u &= \frac{1}{u^\alpha} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \quad (P')$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^N$, $\alpha > 0$. We point out that the existence and uniqueness results are obtained by applying the method of sub and supersolutions without using any approximation argument. our main tools are the Hardy inequality in its simplest form, Lax-Miligram Theorem and a compactness argument in weighted spaces.

Chapter 1

Definitions and General properties

In this chapter, we give some basic concepts that will be used in what follows.

1.1 Lebesgue Spaces

The space $L^p(\Omega)$, where $p \geq 1$, is a Banach space. Its elements are classes of equivalent measurable functions $u(x)$ defined on an open connected set Ω in $\mathbb{R}^n$. (Equivalent functions are functions that differ only on a set of n -dimensional measure 0). The region Ω will be assumed bounded throughout this chapter.

1.1.1 Definitions and Properties of $L^p(\Omega)$ Spaces

Definition 1.1 . Let $1 \leq p < \infty$, we set

$$L^p(\Omega) = \left\{ f : \Omega \longrightarrow \mathbb{R}; f \text{ is measurable and } \int_{\Omega} |f|^p < \infty \right\}$$

with the associated norm

$$\|f\|_{L^p(\Omega)} = \left(\int_{\Omega} |f(x)|^p \right)^{\frac{1}{p}}$$

Definition 1.2 . We define

$$L^{\infty}(\Omega) = \{f : \Omega \rightarrow \mathbb{R} \text{ measurable, } \exists C > 0 : |f(x)| \leq C \text{ a.e. in } \Omega\}$$

With the associated norm

$$\|f\|_{L^{\infty}(\Omega)} = \inf \{C : |f(x)| \leq C \text{ a.e. on } \Omega\}.$$

Remark 1.1 *The space $L^p(\Omega)$ is a Banach space for any $1 \leq p \leq \infty$, reflexive for $1 < p < \infty$, and separable for $1 \leq p \leq \infty$.*

Theorem 1.1 (Lebesgue dominated convergence theorem) *Let (f_n) be a sequence of function in $L^1(\Omega)$ such that*

- $f_n \rightarrow f$ a.e. on Ω ,
 - there is a function $g \in L^1(\Omega)$ such that for all $n : |f_n(x)| \leq g(x)$ a.e. on Ω .
- Then $f \in L^1(\Omega)$ and $\|f_n - f\|_{L^1(\Omega)} \rightarrow 0$.*

Theorem 1.2 (Lebesgue inverse theorem) *If $f_n \rightarrow f$ in $L^p(\Omega)$ then*

- there exist sub sequence $(f_{n,k})$, such that $f_{n,k} \rightarrow f$ a.e. on Ω ,
- $(f_{n,k})$ is upper bounded by $g \in L^p(\Omega)$ i.e.,

$$|f_{n,k}(x)| \leq g(x).$$

1.2 Sobolev Spaces

The $W^{k,p}(\Omega)$ spaces are Banach spaces. The class of functions $W^{k,p}(\Omega)$, can be defined in two ways. specifically, we can define it either as the set of elements $L^p(\Omega)$ possessing generalized derivatives of the first orders in Ω with respect to x_i , for $i = 1, \dots, n$, that are p-summable over Ω , or we can define it as the set of all functions that can be obtained in the form of strong limits.

1.2.1 Generals Definitions

Definition 1.3 *We define Sobolev spaces as follow*

$$W^{k,p}(\Omega) = \{f \in L^p(\Omega); D^\alpha f \in L^p(\Omega), 1 \leq |\alpha| \leq k\}$$

with associated norm

$$\|f\|_{W^{k,p}(\Omega)} = \|f\|_{L^p(\Omega)} + \sum_{\alpha=1}^k \|D^\alpha f\|_{L^p(\Omega)}.$$

We denote by $H^1(\Omega)$ the space $W^{1,2}(\Omega)$.

Remark 1.2 • The space $H^1(\Omega)$ is a Hilbert space with the inner product

$$(u, v)_{H^1(\Omega)} = (u, v)_{L^2(\Omega)} + (\nabla u, \nabla v)_{L^2(\Omega)}.$$

- The space $W^{k,p}(\Omega)$ is Banach space for $1 \leq p \leq \infty$. It is reflexive for $1 < p < \infty$ and separable for $1 \leq p < \infty$. The space $H^1(\Omega)$ is a separable Hilbert space.

Definition 1.4 We define the space $W_0^{k,p}(\Omega)$ as the closure of $\mathcal{D}(\Omega)$ in $W^{k,p}(\Omega)$. We denote by $H_0^1(\Omega)$ the space $W_0^{1,2}(\Omega)$

Remark 1.3 • The space $W_0^{k,p}(\Omega)$ is a banach space with the associated norm:

$$\|f\|_{W_0^{k,p}(\Omega)} = \sum_{\alpha=1}^k \|D^\alpha f\|_{L^p(\Omega)}.$$

- The norm of $W_0^{k,p}(\Omega)$ is equivalent to the norm of $W^{k,p}(\Omega)$.
- The space $W_0^{k,p}(\Omega)$ is separable space. Moreover, it is reflexive for $p > 1$.

Definition 1.5 We define the space $W^{-k,p'}(\Omega)$ as the dual space of $W_0^{k,p}(\Omega)$.

We denote by $W^{-1,2}$ the space $H^{-1}(\Omega)$.

1.2.2 Sobolev Imbedding

Notations Let $1 \leq p \leq \infty$, we denote by p' the sobolev exponent satisfying:

$$\frac{1}{p^*} = \frac{1}{p} - \frac{1}{N},$$

Theorem 1.3 Let $1 \leq p \leq \infty$. We suppose that Ω is bounded open set of class C^1 , or $\Omega = \mathbb{R}_+^N$. We have the following continuous imbedding:

- (i) if $1 \leq p < N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$, $\forall q \in [1, p^*]$,
- (ii) if $p = N$, then $W^{1,p}(\Omega) \hookrightarrow L^q(\Omega)$ $\forall q \in [p, +\infty[$,
- (iii) if $p > N$, then $W^{1,p}(\Omega) \hookrightarrow L^\infty(\Omega)$.

Theorem 1.4 (Rellich-Kondrachov) . We suppose that Ω is bounded open set of class C^1 . We have the following compact imbedding:

- (i) if $1 \leq p < N$, then $W^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$, $\forall q \in [1, p^*]$,
- (ii) if $p = N$, then $W^{1,p}(\Omega) \hookrightarrow_c L^q(\Omega)$, $\forall q \in [p, +\infty[$,
- (iii) if $p > N$, then $W^{1,p}(\Omega) \hookrightarrow_c C(\overline{\Omega})$.

1.3 Some Useful Inequalities

Notations

- Let $1 \leq p < \infty$, we denote by p' the conjugate exponent of p satisfying:

$$\frac{1}{p} + \frac{1}{p'} = 1.$$
- We do note by $d(x) = d(x, \partial\Omega)$ is the distance of x to the boundary $\partial\Omega$.
satisfying: $d(x) = \inf_{y \in \partial\Omega} d(x, y)$.

Proposition 1.5 (Inequality of Poincaré) *Suppose that $1 \leq p < \infty$ and Ω is a bounded open set. Then there exists a constant C (depending only on Ω and p) such that*

$$\|u\|_{L^p(\Omega)} \leq C \|\nabla u\|_{L^p(\Omega)} \quad \forall u \in W_0^{1,p}(\Omega).$$

Proposition 1.6 (Hardy's inequality) *Let Ω be a bounded open set of class C^1 and let $1 < p < \infty$. Set $d(x) = d(x, \partial\Omega)$. There exists a constant c (depending only on Ω) such that*

$$\left\| \frac{f}{d} \right\|_{L^p(\Omega)} \leq c \|\nabla f\|_{L^p(\Omega)} \quad \forall f \in W_0^{1,p}(\Omega).$$

Proposition 1.7 (Holder's Inequality) *Assume that $f \in L^p(\Omega)$ and $g \in L^{p'}(\Omega)$ with $1 \leq p \leq \infty$ then $fg \in L^1(\Omega)$ and*

$$\|fg\|_{L^1(\Omega)} \leq \|f\|_{L^p(\Omega)} \|g\|_{L^{p'}(\Omega)}.$$

For $p = 2$ we obtain the inequality of Cauchy-Schwartz.

Proposition 1.8 (Young's inequality) *For any real numbers a and b we have*

$$|ab| \leq \frac{1}{2}a^2 + \frac{1}{2}b^2.$$

Proposition 1.9 (General Inequality of Young) *For any positive real numbers a and b , we have*

$$ab \leq \varepsilon a^p + C b^{p'}.$$

For any $\varepsilon > 0$ and $p > 1$, where $C > 0$ depends only on ε and p .

1.4 Elliptic Problems and Regularity

Theorem 1.10 (Lax -Milgram Theorem) ([3]) *Let H Hilbert space, assume that*

$$B : H \times H \longrightarrow \mathbb{R}$$

is bilinear mapping, for which there exist constants $\alpha, \beta > 0$ such that

$$|B[u, v]| \leq \alpha \|u\| \|v\| \quad (u, v \in H)$$

and

$$B[u, u] \geq \beta \|u\|^2 \quad (u \in H)$$

Let $f : H \longrightarrow \mathbb{R}$ be a bounded linear functional on H .

Then there exists a unique element $u \in H$ such that

$$B[u, v] = \langle f, v \rangle$$

for all $v \in H$.

Theorem 1.11 (local elliptic regularity) ([5]) *Let D_0 and D be open bounded smooth domains in $\mathbb{R}^N$ with $\overline{D_0} \subset D$, let $p > n$. Then there exists a positive constant $k = k(n, p, \delta(D), D_0, \partial D), L)$ such that for any $w \in W_0^{2,p}(D)$*

$$\|w\|_{W^{2,p}(D_0)} \leq \|\Delta w\|_{L^p(D)} + \|w\|_{L^p(D)}.$$

In particular we have the estimate

$$\|\nabla w\|_{L^\infty(D_0)} \leq \|\Delta w\|_{L^\infty(D)} + \|w\|_{L^\infty(D)}.$$

Lemma 1.12 (Hopf's Lemma) ([10]) *Assume that $u \in \mathcal{C}^2(\Omega) \cap \mathcal{C}^1(\overline{\Omega})$ and*

$$-\Delta u \geq 0 \quad \text{in } \Omega.$$

and there exists a point $x^0 \in \partial\Omega$ such that

$$u(x_0) < u(x), \quad \forall x \in \Omega.$$

Then

$$\frac{\partial u}{\partial n}(x^0) < 0.$$

Chapter 2

Singular Linear Dirichlet Problem

In this chapter we deal with the following linear elliptic problem,

$$\begin{cases} -\Delta u &= f \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega, \quad u > 0 \text{ in } \Omega \end{cases} \quad (P)$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^N$, $f : \Omega \rightarrow \mathbb{R}$ is continuous function, Such that

$$0 \leq f(x) \leq d^{-\beta}(x), \quad \beta > 0.$$

Definition 2.1 *A weak solution of problem (P) is a function $u \in W_0^{1,p}(\Omega)$ and*

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx,$$

$\forall v \in W_0^{1,p'}(\Omega)$, where p depends on β .

2.1 Regularity of Weak Solutions

In this section we will prove the following regularity result for solutions to (P).

Theorem 2.1 *For $0 < \beta < 1$, any weak solution u of problem (P) belongs to $\mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$.*

Theorem 2.2 *For $1 \leq \beta < 2$, any weak solution u of problem (P) belongs to $\mathcal{C}(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$.*

To prove this theorem we recall a useful fact about the Green's function in Ω .

Definition 2.2 *Let Ω be an open domain of $\mathbb{R}^n$. We define the Green function G of $-\Delta$ by the following formula*

$$G(x, y) = \begin{cases} \frac{1}{n(2-n)\omega_n} |x - y|^{2-n}, & n > 2, \\ \frac{1}{2\pi} \log |x - y|, & n = 2. \end{cases}$$

Now we give some Properties of this function.

Proposition 2.3 *Let $u \in C^2(\Omega)$ be the solution of the Dirichlet problem*

$$-\Delta u = f \quad \text{in } \Omega \quad \text{and} \quad u = 0 \quad \text{on } \partial\Omega.$$

Then we have

$$u(x) = \int_{\Omega} (-G(x, y)) f(y) dy, \quad x \in \Omega,$$

Proposition 2.4 . *For $x, y \in \Omega, x \neq y$, the Green's function $G(x, y)$ of $-\Delta$ in Ω verifies:*

$$\begin{aligned} |G(x, y)| &\leq \begin{cases} \frac{c}{|x-y|^{n-2}}, & n > 2, \\ c(|\ln |x-y|| + 1), & n = 2; \end{cases} \\ |G(x, y)| &\leq \begin{cases} \frac{c \, d(y)}{|x-y|^{n-1}}, & n > 2, \\ c \, d(y) \frac{(|\ln |x-y|| + 1)}{|x-y|}, & n = 2; \end{cases} \\ |G_x(x, y)| &\leq \min \left\{ \frac{c}{|x-y|^{n-1}}, \frac{c \, d(y)}{|x-y|^n} \right\}, \quad n \geq 2; \\ |G_{xx}(x, y)| &\leq \min \left\{ \frac{c}{|x-y|^n}, \frac{c \, d(y)}{|x-y|^{n+1}} \right\}, \quad n \geq 2, \end{aligned}$$

where the constant c depends only on Ω .

Proof of Theorem 2.1 by Green's formula, we have

$$u(x) = \int_{\Omega} (-G(x, y)) f(y) dy, \quad x \in \Omega,$$

and

$$\nabla u(x) = \int_{\Omega} (-G_x(x, y)) f(y) dy, \quad x \in \Omega.$$

Then, from the regularity of Ω , there exists $\sigma > 0$ such that for any

$$x_0 \in \Omega_{\sigma} = \{z \in \Omega : d(z) \leq \sigma\}, y \in \Omega,$$

such that

$$d(x_0) = \sigma - |x_0 - y| \text{ and } B_{\sigma(y)} \subset \overline{\Omega}.$$

if $x_1, x_2 \in \Omega$, $\frac{R}{2} = d(x_1, x_2) \leq \sigma$ with $f(x) \leq d(x)^{-\beta}$, we obtain

$$\begin{aligned} |\nabla u(x_1) - \nabla u(x_2)| &\leq \int_{\Omega} |G_x(x_1, y) - G_x(x_2, y)| d(y)^{-\beta} dy \\ &= \int_{\Omega \setminus B_R(x_1)} |G_x(x_1, y) - G_x(x_2, y)| d(y)^{-\beta} dy \\ &\quad + \int_{B_R(x_1)} |G_x(x_1, y) - G_x(x_2, y)| d(y)^{-\beta} dy. \end{aligned}$$

Where we pick

$$I = \int_{\Omega \setminus B_R(x_1)} |G_x(x_1, y) - G_x(x_2, y)| d(y)^{-\beta} dy$$

and

$$II = \int_{B_R(x_1)} |G_x(x_1, y) - G_x(x_2, y)| d(y)^{-\beta} dy.$$

Now we have

$$I \leq \int_{\Omega \setminus B_R(x_1)} \int_0^1 |G_{xx}(\xi(t), y)| |\xi'(t)| d(y)^{-\beta} dy,$$

such that $\xi(t) : [0, 1] \rightarrow \Omega$ with $\xi(t) = x_1 + t(x_2 - x_1)$, So we obtain

$$I \leq \int_{\Omega \setminus B_R(x_1)} \int_0^1 |G_{xx}(\xi(t), y)| |x_2 - x_1| d(y)^{-\beta} dy.$$

Hence

$$I \leq d(x_1, x_2) \int_{\Omega \setminus B_R(x_1)} \int_0^1 |G_{xx}(\xi(t), y)| d(y)^{-\beta} dy,$$

by proposition (2.4) we have

$$I \leq d(x_1, x_2) \int_{\Omega \setminus B_R(x_1)} \int_0^1 \frac{\min\{|\xi(t) - y|, d(y)\}}{|\xi(t) - y|^{n+1}} d(y)^{-\beta} dy$$

$$\leq d(x_1, x_2) \int_{\Omega \setminus B_R(x_1)} \frac{\min\{|x_1 - y|, d(y)\}}{|x_1 - y|^{n+1}} d(y)^{-\beta} dy.$$

Here we have used

$$|\xi(t) - y| \geq |x_1 - y| - |x_1 - \xi(t)|,$$

now we find

$$|x_1 - \xi(t)| = d(x_1, x_2) = \frac{R}{2} \leq \frac{1}{2} |x_1 - y|, \quad \forall y \in \Omega \setminus B_R$$

so

$$|\xi(t) - y| \geq \frac{1}{2} |x_1 - y|$$

We have

$$\begin{aligned} I &\leq cd(x_1, x_2) \int_{\Omega \setminus B_R(x_1)} \frac{1}{|x_1 - y|^{n+\beta}} dy \leq cd(x_1, x_2) \int_R^\infty r^{-n-\beta} r^{n-1} dr \\ I &\leq cd(x_1, x_2) \int_R^\infty r^{-1-\beta} dr = 2cd(x_1, x_2) d^{-\beta}(x_1, x_2) = Cd^{1-\beta}(x_1, x_2). \end{aligned} \quad (2.1)$$

Forthermore, we have

$$II \leq \int_{B_R(x_1)} G_x(x_1, y) d(y)^{-\beta} dy + \int_{B_R(x_2)} G_x(x_2, y) d(y)^{-\beta} dy.$$

as above by the proposition 2.4 we obtain

$$\int_{B_R(x_1)} G_x(x_1, y) d(y)^{-\beta} dy \leq c \int_{B_R(x_1)} \frac{\min\{|x_1 - y|, d(y)\}}{|x_1 - y|^n} d(y)^{-\beta} dy.$$

and

$$\int_{B_R(x_2)} G_x(x_2, y) d(y)^{-\beta} dy \leq c \int_{B_R(x_2)} \frac{\min\{|x_2 - y|, d(y)\}}{|x_2 - y|^n} d(y)^{-\beta} dy.$$

As above

$$II \leq \int_{B_R(x_1)} |x_1 - y|^{-n-\beta+1} dy + \int_{B_R(x_2)} |x_2 - y|^{-n-\beta+1} dy$$

so

$$II \leq \int_0^R r^{-n-\beta+1} r^{n-1} dr = 2cd^{1-\beta}(x_1, x_2). \quad (2.2)$$

By (2.1) and (2.2), we can easily conclude Theorem 2.1.

Proof of Theorem 2.2 by Green's formula, we have

$$u(x) = \int_{\Omega} (-G(x, y)) f(y) dy, \quad x \in \Omega,$$

and

$$\nabla u(x) = \int_{\Omega} (-G_x(x, y)) f(y) dy, \quad x \in \Omega.$$

Then, if $x_1, x_2 \in \Omega$, $R = 2d(x_1, x_2)$ with $f(x) \leq d(x)^{-\beta}$, we obtain

$$\begin{aligned} |u(x_1) - u(x_2)| &\leq \int_{\Omega} |G(x_1, y) - G(x_2, y)| d(y)^{-\beta} dy \\ &= \int_{\Omega \setminus B_R(x_1)} |G(x_1, y) - G(x_2, y)| d(y)^{-\beta} dy \\ &\quad + \int_{B_R(x_1)} |G(x_1, y) - G(x_2, y)| d(y)^{-\beta} dy. \end{aligned}$$

We pick

$$I = \int_{\Omega \setminus B_R(x_1)} |G(x_1, y) - G(x_2, y)| d(y)^{-\beta} dy$$

and

$$II = \int_{B_R(x_1)} |G(x_1, y) - G(x_2, y)| d(y)^{-\beta} dy.$$

Now we find

$$I \leq \int_{\Omega \setminus B_R(x_1)} \int_0^1 |G_x(\xi(t), y)| |\xi'(t)| d(y)^{-\beta} dy,$$

such that $\xi(t) : [0, 1] \rightarrow \Omega$ with $\xi(t) = x_1 + t(x_2 - x_1)$, So we obtain

$$I \leq \int_{\Omega \setminus B_R(x_1)} \int_0^1 |G_x(\xi(t), y)| |x_2 - x_1| d(y)^{-\beta} dy.$$

Hence

$$I \leq d(x_1, x_2) \int_{\Omega \setminus B_R(x_1)} \int_0^1 |G_x(\xi(t), y)| d(y)^{-\beta} dy,$$

by proposition (2.4) we have

$$\begin{aligned} I &\leq d(x_1, x_2) \int_{\Omega \setminus B_R(x_1)} \int_0^1 \frac{\min\{|\xi(t) - y|, d(y)\}}{|\xi(t) - y|^n} d(y)^{-\beta} dy \\ &\leq d(x_1, x_2) \int_{\Omega \setminus B_R(x_1)} \frac{\min\{|x_1 - y|, d(y)\}}{|x_1 - y|^n} d(y)^{-\beta} dy. \end{aligned}$$

so

$$I \leq cd(x_1, x_2) \int_{\Omega \setminus B_R(x_1)} \frac{1}{|x_1 - y|^{n-1+\beta}} dy \leq cd(x_1, x_2) \int_R^\infty r^{-\beta} dr = Cd^{2-\beta}(x_1, x_2). \quad (2.3)$$

Furthermore, as above we have

$$II \leq \int_{B_R(x_1)} \int_0^1 |G_x(\xi(t), y)| |\xi'(t)| d(y)^{-\beta} dy$$

and

$$\leq d(x_1, x_2) \int_{B_R(x_1)} \frac{\min\{|x_1 - y|, d(y)\}}{|x_1 - y|^n} d(y)^{-\beta} dy.$$

By the same way we find

$$II \leq cd(x_1, x_2) \int_{B_R(x_1)} |x_1 - y|^{1-n-\beta} dy,$$

so

$$II \leq cd(x_1, x_2) \int_{B_R(x_1)} r^{1-n-\beta} r^{n-1} dr = cd(x_1, x_2) \int_0^R r^{-\beta} dr = Cd^{2-\beta}(x_1, x_2). \quad (2.4)$$

By (2.3) and (2.4), we can easily conclude Theorem 2.2.

2.2 Existence and Uniqueness of Solutions

In this section we will study the existence of solutions for the problem (P).

Theorem 2.5 *Let $0 < \beta < \frac{3}{2}$. Then there exist a unique solution $u \in H_0^1(\Omega)$ to problem (P)*

Proof. Indeed, we apply the Lax-Milgram theorem. The associated bilinear form is

$$a(u, v) = \int_{\Omega} \nabla u \nabla v dx$$

Now we have to prove the continuity of $a(u, v)$. By using the inequality of Cauchy-Schwarz we obtain

$$|a(u, v)| \leq \int_{\Omega} |\nabla u \nabla v| dx \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)}.$$

Now we prove that $a(u, v)$ is coercive

$$a(u, u) = \int_{\Omega} |\nabla u|^2 dx = \|u\|^2.$$

It remains only to prove that $f \in H^{-1}(\Omega)$, we have

$$|\langle f, v \rangle| = \left| \int_{\Omega} f v dx \right| \leq \left| \int_{\Omega} \frac{v}{d^{\beta}} \frac{d}{d} dx \right| = \left| \int_{\Omega} \frac{v}{d} d^{1-\beta} dx \right|.$$

By using Cauchy-Schwarz inequality, we obtain

$$\left| \int_{\Omega} \frac{v}{d} d^{1-\beta} dx \right| \leq \left(\int_{\Omega} \left| \frac{v}{d} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |d^{1-\beta}|^2 dx \right)^{\frac{1}{2}}.$$

Now by using Hardy's inequality, we find

$$\left(\int_{\Omega} \left| \frac{v}{d} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |d^{1-\beta}|^2 dx \right)^{\frac{1}{2}} \leq \left(c \int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}} = c \|v\|.$$

Since

$$c = \left(\int_{\Omega} |d^{1-\beta}|^2 dx \right)^{\frac{1}{2}} < \infty$$

(we have in fact $2(1-\beta) > -1$).

So by Lax-Milgram theorem there exists a unique solution $u \in H_0^1(\Omega)$ to the problem (P).

Theorem 2.6 *Let $\frac{3}{2} \leq \beta < 2$. Then there exist a unique solution $u \in W_0^{1,p}(\Omega)$ for $1 < p < \frac{1}{\beta-1}$ to the problem (P).*

To prove the last theorem. First, we need to prove the following main theorem .

Theorem 2.7 *Let $\frac{3}{2} \leq \beta < 2$ and let u be a weak solution to the problem (P), then there exist $c > 0$ such that:*

$$|\nabla u(x)| \leq c d(x)^{1-\beta} \quad \forall x \in \Omega.$$

To prove the above theorem, we begin by the following lemma.

Lemma 2.8 *There exists a constant $k_1 > 0$ such that, if $v \in L^\infty(\Omega)$, $-\Delta v \in L^\infty(K)$, for all compact subset K of Ω and if $r \in (0, 1]$, $B_{2r}(x_0) \subset \Omega$, we have*

$$\forall x \in B_r(x_0) : |\nabla v(x)| \leq k_1 \left(r \|\Delta v\|_{L^\infty(B_{2r}(x_0))} + \frac{1}{r} \|v\|_{L^\infty(B_{2r}(x_0))} \right) \quad (2.5)$$

Proof. Let $x_0 \in \Omega$, and let $r : 0 < 2r < d(x_0)$ (then $B_{2r}(x_0) \subset \Omega$). Changing variables according to $x_0 + ry = x$, we define $w(y) = v(x)$, for $y \in B_2(0)$. Then we have

$$\nabla w(y) = r \nabla v(x), \quad \Delta w(y) = r^2 \Delta v(x) \quad (2.6)$$

and we use the following estimate followed from the local regularity (Theorem 1.11),

$$|\nabla w(y)| \leq k \left(\|\Delta w\|_{L^\infty(B_2(0))} + \|w\|_{L^\infty(B_2(0))} \right), \text{ for all } y \in B_1(0) \quad (2.7)$$

Hence, the local estimate (2.5) follows from (2.6) and (2.7).

Proof of Theorem 2.7. Let $x \in \Omega$ and set $r = \frac{d(x)}{3}$, and take $x_0 = x$. Let us note that

$$B_{2r}(x) \subset A = \left\{ z \in \Omega : \frac{d(x)}{3} \leq d(z) \leq \frac{5}{3}d(x) \right\} \subset \Omega.$$

We apply the previous lemma, we obtain

$$|\nabla u(x)| \leq c \left(\|\Delta u\|_{L^\infty(A)} + \frac{1}{d(x)} \|u\|_{L^\infty(A)} \right).$$

In other side, we can prove easily that $u_0 = c_1 \phi_1^{2-\beta}$ is a subsolution of the problem (P) for c_1 small enough, that is

$$-\Delta u_0 - f \leq c_1 d^{-\beta} - c d^{-\beta} \leq 0,$$

and $u^0 = C d(x)^{2-\beta}$ is a supersolution for c_2 large enough. that is

$$-\Delta u^0 - f \geq c_2 d^{-\beta} - c d^{-\beta} \geq 0.$$

Thus

$$c_1 d(x)^{2-\beta} \leq u \leq c_2 d(x)^{2-\beta}.$$

This gives

$$|\nabla u(x)| \leq c \left(d(x) \frac{d^{-\beta}}{3^{-\beta}} + \frac{1}{d(x)} \frac{d(x)^{2-\beta}}{3^{2-\beta}} \right) \leq C d^{1-\beta}.$$

Proof of theorem 2.6 First we have to prove that $f \in W^{-1,p}(\Omega)$, for $1 < p < \frac{1}{\beta-1}$, we have

$$|\langle f, v \rangle| = \left| \int_{\Omega} f v \, dx \right| \leq \left| \int_{\Omega} \frac{v}{d^\beta} \frac{d}{d} dx \right| = \left| \int_{\Omega} \frac{v}{d} d^{1-\beta} dx \right|.$$

By using Holder's inequality, we obtain

$$\left| \int_{\Omega} \frac{v}{d} d^{1-\beta} dx \right| \leq \left(\int_{\Omega} \left| \frac{v}{d} \right|^{p'} dx \right)^{\frac{1}{p'}} \left(\int_{\Omega} |d^{1-\beta}|^p dx \right)^{\frac{1}{p}}.$$

Now by using Hardy's inequality, we find

$$\left(\int_{\Omega} \left| \frac{v}{d} \right|^{p'} dx \right)^{\frac{1}{p'}} \left(\int_{\Omega} |d^{1-\beta}|^p dx \right)^{\frac{1}{p}} \leq \left(c \int_{\Omega} |\nabla v|^{p'} dx \right)^{\frac{1}{p'}} = c \|v\|_{W_0^{1,p'}(\Omega)}.$$

Where

$$c = \left(\int_{\Omega} |d^{1-\beta}|^p dx \right)^{\frac{1}{p}} < \infty,$$

since

$$1 < p < \frac{1}{\beta - 1} \implies p(1 - \beta) > -1.$$

Now we have the following auxiliary problem,

$$\begin{cases} -\Delta u_{\epsilon} &= f_{\epsilon} \text{ in } \Omega \\ u_{\epsilon} &= 0 \text{ on } \partial\Omega, \quad u_{\epsilon} > 0 \text{ in } \Omega \end{cases}$$

where

$f_{\epsilon} \in L^{\infty}(\Omega)$, f is continuous function, $f_{\epsilon} \longrightarrow f$ uniformly on the compact subset of Ω , ■

which gives

$$f_{\epsilon} \longrightarrow f \text{ in } W^{-1,p}(\Omega), \text{ as } \epsilon \rightarrow 0.$$

So by lax-Milgram theorem there exist a unique solution $u_{\epsilon} \in H_0^1(\Omega)$ to the last problem, by the theorem 2.1 $u_{\epsilon} \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$. this gives $u_{\epsilon} \in W_0^{1,p}(\Omega)$. In other side, by the theorem 2.7

$$\|\nabla u_{\epsilon}\|_{L^p(\Omega)} \leq c \|d^{1-\beta}\|_{L^p(\Omega)} \leq C,$$

this gives $u_{\epsilon} \longrightarrow u$ weakly in $W_0^{1,p}(\Omega)$, so we can passe to the limit as $\epsilon \rightarrow 0$, in the weak formulation

$$\int_{\Omega} \nabla u_{\epsilon} \nabla v dx = \int_{\Omega} f_{\epsilon} v dx.$$

we obtain $\int_{\Omega} \nabla u_{\epsilon} \nabla v dx \longrightarrow \int_{\Omega} \nabla u \nabla v dx$, and $\int_{\Omega} f_{\epsilon} v dx \longrightarrow \int_{\Omega} f v$, thus u is a weak solution to the problem (P) moreover $u \in W_0^{1,p}(\Omega)$. It remains only to prove that u is unique, let u, v two solutions to the last problem, we have $-\Delta(u - v) = 0$ which implies $u - v = 0$ and this gives $u = v$. So we can easily conclude the Theorem 2.6.

2.3 Non-existence Result for $\beta \geq 2$

In this section we prove that the following problem

$$\begin{cases} -\Delta u &= \frac{1}{d^\beta} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \quad (2.8)$$

has no weak solution.

Theorem 2.9 *For $\beta \geq 2$, the problem (2.8) has no solutions $u \in \mathcal{C}(\overline{\Omega}) \cap W_0^{1,p}(\Omega)$ for any $p \geq 1$.*

To prove this theorem we need the following lemma.

Lemma 2.10 *Let $\psi \in \mathcal{C}^1(\overline{\Omega}) \cap \mathcal{C}^2(\Omega)$ to be the unique solution to the linear problem*

$$-\Delta\psi = 1 \text{ in } \Omega, \quad \psi = 0 \text{ on } \partial\Omega.$$

Then

$$\psi \sim d.$$

Proof. First we work with $y \in \partial\Omega$, we apply the mean value theorem, we obtain

$$|\psi(x) - \psi(y)| \leq c|x - y|,$$

with $y \in \partial\Omega$ we find,

$$|\psi(x)| \leq cd(y). \quad (2.9)$$

Now, by the theorem 2.1, we can apply the Hopf lemma, we obtain $|\nabla\psi| > 0$ on $\partial\Omega$, since Ω is bounded and regular, we obtain

$$|\nabla\psi| \geq c > 0, \quad \text{near to } \partial\Omega.$$

Let $y \in \Omega$ (near to $\partial\Omega$) and let n be the internal unit normal vector on $\partial\Omega$ at y . we have

$$|\nabla\psi(y)| = \lim_{t \rightarrow 0} \frac{|\psi(y - tn) - \psi(y)|}{t} = \lim_{t \rightarrow 0} \frac{|\psi(y - tn)|}{t} \geq c.$$

so for $0 < t < \eta$, we have

$$\frac{|\psi(y - tn)|}{t} \geq c \implies \psi(y - tn) \geq ct.$$

For $x = y - tn$ we obtain

$$\psi(x) \geq c|x - y| \geq cd(x) \quad \forall x \in \{x \in \Omega : d(x) \leq \eta\}. \quad (2.10)$$

Now we work with y in the interior of Ω , we have

$$c_1 \leq d(x) \leq c_2$$

and

$$c'_1 \leq \psi(x) \leq c'_2,$$

then

$$\psi(x) \geq \frac{c'}{c_2} c_2 \geq Cd(x). \quad (2.11)$$

By (2.9), (2.10) and (2.11), we obtain

$$\psi \sim d.$$

Proof of theorem 2.9 Suppose by contradiction that there exists a solution $u_0 \in C(\bar{\Omega}) \cap W_0^{1,p}(\Omega)$ of problem (2.8). We consider the following auxiliary problem

$$\begin{cases} -\Delta u_\epsilon &= \frac{1}{(d+\epsilon)^\beta} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \quad (2.12)$$

By the comparison principle we can prove that,

$$c\psi \leq u_\epsilon \leq u_0 \quad \text{in } \Omega. \quad (2.13)$$

Let λ_1 be the first eigenvalue with the associated eigenfunction φ_1 of $-\Delta$ in Ω , it is well known that $\lambda_1 > 0$ and $\varphi_1 \in \mathcal{C}^2(\Omega)$, by the same proof of lemma 2.10 we obtain

$$c_1 d \leq \varphi_1 \leq c_2 d. \quad (2.14)$$

Now multiplying (2.12) by φ_1 and integrating by parts we obtain

$$\lambda_1 \int_{\Omega} u_\epsilon \varphi_1 dx = \int_{\Omega} (d + \epsilon)^{-\beta} \varphi_1 dx,$$

by using (2.13) we obtain

$$M = \lambda \int_{\Omega} u^0 \varphi_1 dx \geq \lambda \int_{\Omega} u_\epsilon \varphi_1 dx = \int_{\Omega} (d + \epsilon)^{-\beta} \varphi_1 dx.$$

Passing to the limit as $\epsilon \rightarrow 0$ in the above inequality and using (2.14) we find

$$M \geq \int_{\Omega} d^{1-\beta} dx = \int_{\Omega} d^{1-\beta} dx,$$

but $\int_{\Omega} d^{1-\beta} dx = \infty$ for $\beta \geq 2$, which is contradiction. This completes the proof.

Chapter 3

Existence and uniqueness of Singular Problem

In this chapter we study the existence and uniqueness of positive weak solution to the elliptic quasilinear and singular problem

$$\begin{cases} -\Delta u &= \frac{1}{u^\alpha} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \quad (P')$$

where Ω is an open bounded domain with smooth boundary in $\mathbb{R}^N$, $\alpha > 0$.

The problem (P') is reduced to an equivalent fixed point problem which is studying by using a method of sub and supersolutions giving rise to monotone sequences converging to fixed points which are actually minimal and maximal solutions (which may coincide) in the interval between the ordered sub and supersolutions.

Definition 3.1 Let $p \geq 1$, for the weight function $b(x)$ we define

$$L^p(\Omega, b) = \left\{ u \in L^p(\Omega) : \int_{\Omega} |u(x)|^p b(x) dx < \infty \right\}$$

With the associated norm

$$\|u\|_{L^p(\Omega, b)} = \left(\int_{\Omega} |u|^p b(x) dx \right)^{\frac{1}{p}}.$$

Definition 3.2 We say that u_0 (resp. u^0) is a subsolution (resp. a supersolution) of problem (P') , if u_0, u^0 belong to $H_0^1(\Omega) \cap L^\infty(\Omega)$ and

$$\int_{\Omega} \nabla u_0 \nabla v dx - \int_{\Omega} (u_0^{-\alpha}) v dx \leq 0 \quad (3.1)$$

and

$$(resp. \int_{\Omega} \nabla u^0 \nabla v dx - \int_{\Omega} (u^0)^{-\alpha} v dx \geq 0) \quad (3.2)$$

$$\forall v \in H_0^1(\Omega), v \geq 0.$$

Definition 3.3 We say that $\underline{u}$ (resp. $\bar{u}$) is a minimal solution (resp. a maximal solution) of problem (P') if $\underline{u}$ is weak solution and satisfying,

$$\underline{u} \leq v^0, \quad \text{for any supersolution } v^0,$$

$$(resp. v_0 \leq \bar{u}, \quad \text{for any subsolution } v_0.)$$

3.1 Existence for The case $0 < \alpha < 1$

We study the existence and uniqueness of positive weak solution to the problem (P') , where $0 < \alpha < 1$.

Definition 3.4 a weak solution of problem (P') is a function $u \in H_0^1(\Omega)$ and

$$\int_{\Omega} \nabla u \nabla v dx = \int_{\Omega} f v dx,$$

$$\forall v \in H_0^1(\Omega).$$

Our goal in this section is to prove the following theorem

Theorem 3.1 The problem (P') has a positive weak solution u satisfying,

$$c_1 d(x) \leq u(x) \leq c_2 d(x).$$

Therefor our main tool is the proof of following existence theorem.

Theorem 3.2 Assume that there exists a subsolution u_0 and supersolution u^0 such that $u_0 \leq u^0$ and that there exist constants c_1, c_2 satisfying:

$$0 < c_1 d(x) \leq u_0(x) \leq u^0(x) \leq c_2 d(x) \quad \forall x \in \Omega.$$

Then, there exists a minimal solution $\underline{u}$ and maximal solution $\bar{u}$ to the problem (P') such that

$$u_0 \leq \underline{u} \leq \bar{u} \leq u^0.$$

We define the following weight function $b(x) = \frac{1}{d(x)^{1+\alpha}}$ and the sub set

$$K = [u_0, u^0] = \{u \in L^2(\Omega, b) : u_0 \leq u \leq u^0\}.$$

Notice that K is convex, closed and bounded. A first auxiliary result which is the following.

Lemma 3.3 *There exists a positive constant $M > 0$ such that the mapping $F : K \rightarrow H^{-1}(\Omega)$ defined by $F(w) = \frac{1}{w^\alpha} + M \frac{w}{d(x)^{1+\alpha}}$ is well defined, continuous and increasing.*

Proof. To prove that F is well defined we have to prove that $F(w) \in H^{-1}(\Omega)$, $\forall w \in K$, the linearity is immediate, so we have to prove the continuity.

Let $z \in H_0^1(\Omega)$ by using the fact that $w \in K$, we obtain for the first term of $F(w)$

$$\left| \left\langle \frac{1}{w^\alpha}, z \right\rangle \right| = \left| \int_{\Omega} \frac{z}{w^\alpha} dx \right| \leq c \int_{\Omega} \left| \frac{z}{d^\alpha} \right| d^{1-\alpha} dx$$

by using inequality of Cauchy-Schwarz and Hardy's inequality we obtain

$$c \int_{\Omega} \left| \frac{z}{d} \right| d^{1-\alpha} dx \leq c \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \|d^{1-\alpha}\|_{L^2(\Omega)} \leq c \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \leq c \|z\|.$$

Where c denotes different positive constants which are independent of the function involved, in the same vein, we denote by $\|z\|$ the norm $(\int_{\Omega} |\nabla z|^2 dx)^{\frac{1}{2}}$ in the Sobolev space $H_0^1(\Omega)$ (as along this chapter).

For the second term of $F(w)$ we have for all $z \in H_0^1(\Omega)$ and using inequality of Cauchy-Schwarz and Hardy's inequality we obtain,

$$\left| \left\langle \frac{w}{d^{1+\alpha}}, z \right\rangle \right| = \left| \int_{\Omega} \frac{zw}{d^{1+\alpha}} dx \right| \leq \int_{\Omega} \left| \frac{z}{d} \right| \left| \frac{w}{d^\alpha} \right| dx \leq \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \left\| \frac{w}{d^\alpha} \right\|_{L^2(\Omega)} \leq c \|z\|$$

where the constant $c > 0$ is given by :

$$\left\| \frac{w}{d^\alpha} \right\|_{L^2(\Omega)}^2 = \int_{\Omega} \left| \frac{w^2}{d^{2\alpha}} \right| dx = \int_{\Omega} \frac{w^2}{d^{1+\alpha}} d^{1-\alpha} dx \leq c \|w\|_{L^2(\Omega, b)}.$$

Nexte we prove the continuity of F . For the first term, if we assume that $w_n \rightarrow w$ in $L^2(\Omega, b)$, we should prove that

$$\left\| \frac{1}{w_n^\alpha} - \frac{1}{w^\alpha} \right\|_{H^{-1}(\Omega)} \longrightarrow 0 \quad \text{as } n \rightarrow \infty$$

we have

$$\left| \int_{\Omega} \left(\frac{1}{w_n^\alpha} - \frac{1}{w^\alpha} \right) z dx \right| = \left| \int_{\Omega} \frac{w^\alpha - w_n^\alpha}{w^\alpha w^\alpha} \left(\frac{z}{d} \right) d dx \right| \leq c'_n \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \leq c'_n \|z\| dx.$$

Where

$$c'_n = \left\| \frac{d(w_n^\alpha - w^\alpha)}{w_n^\alpha w^\alpha} \right\|_{L^2(\Omega)}^2$$

Now using the mean value theorem we find

$$\begin{aligned} c'_n &= \left\| \frac{d(w_n - w) \alpha w(\theta)^{\alpha-1}}{w_n^\alpha w^\alpha} \right\|_{L^2(\Omega)}^2 \\ &= \int_{\Omega} \left| \frac{d^2(w_n - w)^2 \alpha^2 w(\theta)^{2(\alpha-1)}}{w_n^2 \alpha w^2 \alpha} \right| dx \end{aligned}$$

and by the definition of K

$$\begin{aligned} c'_n &\leq c \int_{\Omega} \frac{|w_n - w|^2 d^{2(\alpha-1)} \alpha^2}{|w|^{4\alpha}} dx \\ &\leq c' \int_{\Omega} \frac{|w_n - w|^2 d^{2\alpha}}{d^4 \alpha} dx \\ &\leq c' \int_{\Omega} \frac{|w_n - w|^2}{d^{2\alpha}} dx \\ &\leq c' \int_{\Omega} \frac{|w_n - w|^2}{d^{1+\alpha}} dx \\ &\leq c' \|w_n - w\|_{L^2(\Omega, b)}^2 \end{aligned}$$

which converges to 0 as $n \rightarrow \infty$ (here θ do notes the intermediate point in in the segment $[w_n, w]$). For the second term of F , we have for any $z \in H_0^1(\Omega)$

$$\begin{aligned} \left| \int_{\Omega} \left(\frac{w_n}{d^{1+\alpha}} - \frac{w}{d^{1+\alpha}} \right) z dx \right| &\leq \int_{\Omega} \left| \frac{w_n - w}{d^{1+\alpha}} z \right| dx \\ &\leq \int_{\Omega} \left| \frac{w_n - w}{d^{1+\alpha}} \right| |z| dx \\ &\leq \int_{\Omega} \left| \frac{w_n - w}{d^\alpha} \right| \left| \frac{z}{d} \right| dx, \end{aligned}$$

by the inequality of Couchy-Shwarz

$$\int_{\Omega} \left| \frac{w_n - w}{d^\alpha} \right| \left| \frac{z}{d} \right| dx \leq \left(\int_{\Omega} \left| \frac{w_n - w}{d^\alpha} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| \frac{z}{d} \right|^2 dx \right)^{\frac{1}{2}}$$

by Hardy's inequality

$$\left| \int_{\Omega} \left(\frac{w_n}{d^{1+\alpha}} - \frac{w}{d^{1+\alpha}} \right) z dx \right| \leq \int_{\Omega} \frac{|w_n - w|^2}{d^{2\alpha}} \int_{\Omega} |\nabla z|^2 dx$$

we have now

$$\int_{\Omega} \frac{|w_n - w|^2}{d^{2\alpha}} dx = \int_{\Omega} \frac{|w_n - w|^2}{d^{1+\alpha}} d^{1-\alpha} dx \leq c \|w_n - w\|_{L^2(\Omega, b)}^2$$

from where we obtain

$$\left| \left\langle \frac{w_n - w}{d^{1+\alpha}}, z \right\rangle \right| \leq c \|w_n - w\|_{L^2(\Omega, b)}^2 \|z\|.$$

Which converges to 0 as $n \rightarrow \infty$.

Now we prove that F is increasing, we work in the bounded interval $[u, v] \subset K$, using the definition of K , the mean value theorem and choose the constant M large.

$$\begin{aligned} F(u) - F(v) &= \frac{1}{u^\alpha} + M \frac{u}{d^{1+\alpha}} - \frac{1}{v^\alpha} - M \frac{v}{d^{1+\alpha}} \\ &= -\alpha w(\theta)^{-\alpha-1} (u - v) - \frac{M}{d^{1+\alpha}} (u - v) \\ &= \left(\frac{-M}{d^{1+\alpha}} - \alpha w^{-\alpha-1} \right) (u - v) \\ &\leq (M - \alpha) \left(\frac{u-v}{d^{1+\alpha}} \right) \leq 0 \end{aligned} \tag{3.3}$$

(here θ do notes the intermediate point in in the segment $[u, v]$).

We need a second result which is the following.

Lemma 3.4 *If $0 < \alpha < 1$, for any $h \in H^{-1}(\Omega)$, there exists a unique solution $z \in H_0^1(\Omega)$, to the linear problem*

$$\begin{cases} -\Delta z + \frac{Mz}{d(x)^{1+\alpha}} = h & \text{in } \Omega \\ z = 0 & \text{on } \partial\Omega \end{cases} \tag{3.4}$$

Proof. Indeed we apply the Lax-Milgram theorem. The associated bilinear form is

$$a(u, v) = \int_{\Omega} \nabla u \nabla v dx + M \int_{\Omega} \frac{uv}{d(x)^{1+\alpha}} dx$$

Now we have to prove the continuity of $a(u, v)$, we have

$$|a(u, v)| \leq \int_{\Omega} |\nabla u \nabla v| dx + M \int_{\Omega} \left| \frac{u}{d} \right| \left| \frac{v}{d} \right| d^{1-\alpha} dx$$

By using the inequality of Cauchy-Schwarz we obtain

$$|a(u, v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + c \left\| \frac{u}{d} \right\|_{L^2(\Omega)} \left\| \frac{v}{d} \right\|_{L^2(\Omega)}, c = \int_{\Omega} d^{1-\alpha} dx < \infty.$$

We apply Hardy's inequality for the second term we find

$$|a(u, v)| \leq \|u\| \|v\| + c \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq (c + 1) \|u\| \|v\|.$$

Now we prove that $a(u, v)$ is coercive

$$a(u, u) = \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} M \frac{u^2}{d^{1+\alpha}} dx \geq \int_{\Omega} |\nabla u|^2 dx = \|u\|.$$

With $h \in H^{-1}(\Omega)$ so by Lax-Milgram theorem there exists a unique solution $u \in H_0^1(\Omega)$ of (3.4). We need a final auxiliary lemma which is the following.

Lemma 3.5 *The imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega, c)$ is compact, where $c(x) = \frac{1}{d(x)^\beta}$, for $\beta < 2$.*

Proof. In the first we have to prove that $H_0^1(\Omega) \subset L^2(\Omega, c)$. Let $u \in H_0^1(\Omega)$, by Hardy's inequality, we have

$$\int_{\Omega} \frac{|u|^2}{d^\beta} dx = \int_{\Omega} \frac{|u|^2}{d^\beta} \frac{d^2}{d^2} dx \leq c \int_{\Omega} \left| \frac{u}{d} \right|^2 dx \leq c \int_{\Omega} |\nabla u|^2 = c \|u\| < \infty. \quad (3.5)$$

So $u \in L^2(\Omega, c)$. Now taking a sequence $u_n \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and prove that $u_n \rightarrow u$ strongly in $L^2(\Omega, c)$ as $n \rightarrow +\infty$, we obtain

$$\|u_n - u\|_{L^2(\Omega, c)} = \int_{\Omega} \frac{|(u_n - u)|^2}{d^\beta} dx = \int_{\Omega} |(u_n - u)|^{2-\beta} \frac{(u_n - u)^\beta}{d^\beta} dx$$

by Holder's inequality

$$\|u_n - u\|_{L^2(\Omega, c)} \leq \left(\int_{\Omega} |(u_n - u)^{2-\beta}|^p dx \right)^{\frac{1}{p}} \left(\int_{\Omega} \left| \left(\frac{u_n - u}{d} \right)^\beta \right|^{p'} dx \right)^{\frac{1}{p'}}$$

we have to take $p = \frac{2}{2-\beta}$ and $p' = \frac{2}{\beta}$

$$\int_{\Omega} \frac{|(u_n - u)|^2}{d^{\beta}} dx \leq \int_{\Omega} (u_n - u)^2 dx \int_{\Omega} \left(\frac{u}{d}\right)^2 dx$$

as $H_0^1(\Omega) \hookrightarrow_c L^2(\Omega)$ so $\int_{\Omega} (u_n - u)^2 dx \rightarrow 0$ and $\int_{\Omega} \left(\frac{u}{d}\right)^2 dx$ is bounded, this gives

$$\int_{\Omega} \frac{|u_n - u|^2}{d^{\beta}} dx \rightarrow 0,$$

so

$$H_0^1(\Omega) \hookrightarrow_c L^2(\Omega, c).$$

Proof of Theorem 3.2 Problem (P') is obviously equivalent to the nonlinear problem

$$\begin{cases} -\Delta u + M \frac{u}{d(x)^{1+\alpha}} &= \frac{1}{u^{\alpha}} + M \frac{u}{d(x)^{1+\alpha}} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.6)$$

By the definition of the subsolutions, we have

$$\int_{\Omega} \nabla u_0 \nabla v dx - \int_{\Omega} (u_0^{-\alpha}) v dx \leq 0 \quad (3.7)$$

by Lemma 3.4 and Lemma 3.3 we construct the following iterative scheme starting from the subsolution u_0 (and it is very similar for supersolution u^0), such that $u_{k+1} \in H_0^1(\Omega)$ to be unique weak solution of the linear boundary value problem.

$$\begin{cases} \Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^{1+\alpha}} &= \frac{1}{u_k^{\alpha}} + M \frac{u_k}{d(x)^{1+\alpha}} \\ u_{k+1} &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.8)$$

We have to prove that

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq u^k \leq \dots \leq u^1 \leq u^0.$$

First we confirm this for $k = 0$ in (3.8), we have

$$\begin{cases} \Delta u_1 + M \frac{u_1}{d(x)^{1+\alpha}} &= \frac{1}{u_0^{\alpha}} + M \frac{u_0}{d(x)^{1+\alpha}} \text{ in } \Omega \\ u_1 &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.9)$$

Subtract (3.9) from (3.7), multiplying by $v = (u_0 - u_1)^+ \in H_0^1(\Omega)$, $v \geq 0$ and integrating by parts, we obtain

$$\int_{\Omega} \nabla (u_0 - u_1)^{+2} dx + \int_{\Omega} M \frac{u_1}{d(x)^{1+\alpha}} (u_0 - u_1)^{+2} dx \leq 0 \quad (3.10)$$

so $\int_{\Omega} \nabla(u_0 - u_1)^+ dx = 0$ which implies $u_0 \leq u_1$. Now we show

$$u_{k-1} \leq u_k \implies u_k \leq u_{k+1} \quad (3.11)$$

from (3.8) for each $v \in H_0^1(\Omega)$ we find

$$\int_{\Omega} \nabla u_{k+1} \nabla v + M \frac{u_{k+1}}{d(x)^{1+\alpha}} v dx = \int_{\Omega} \left(\frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^{1+\alpha}} \right) v dx$$

and

$$\int_{\Omega} \nabla u_k \nabla v + M \frac{u_k}{d(x)^{1+\alpha}} v dx = \int_{\Omega} \left(\frac{1}{u_{k-1}^\alpha} + M \frac{u_{k-1}}{d(x)^{1+\alpha}} \right) v dx$$

subtract the preceding formulas and take $v = (u_k - u_{k-1})^+$, we obtain

$$\int_{\Omega} (\nabla(u_k - u_{k+1})^+)^2 + \frac{Mu_1}{d(x)^{1+\alpha}} (u_k - u_{k+1})^+ dx = \int_{\Omega} (F(u_{k-1}) - F(u_k)) (u_k - u_{k+1})^+ dx,$$

by lemma 3.3 and 3.11

$$F(u_{k-1}) - F(u_k) \leq 0 \implies u_k \leq u_{k+1}.$$

Next we assume that $u_k \leq u^0$ and prove that $u_{k+1} \leq u^0$

$$\begin{aligned} -\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^{1+\alpha}} &= \frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^{1+\alpha}} \\ -\Delta u^0 + M \frac{u^0}{d(x)^{1+\alpha}} &\geq \frac{1}{(u^0)^\alpha} + M \frac{u^0}{d(x)^{1+\alpha}}. \end{aligned}$$

Subtract the preceding formulas, multiplying by $v = (u_{k+1} - u^0)^+ \in H_0^1(\Omega)$, integrating by parts and by lemma 3.3, we find

$$\begin{aligned} \int_{\Omega} (\nabla(u_{k+1} - u^0)^+)^2 + M \frac{1}{d(x)^{1+\alpha}} (u_{k+1} - u^0)^+ dx &\leq \int_{\Omega} (F(u_k) - F(u^0)) (u_{k+1} - u^0)^+ dx \\ &\leq 0 \end{aligned}$$

this gives $u_{k+1} \leq u^0$, thus we have established the increasing sequence;

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq u^0.$$

By the same way we establish second decreasing sequence;

$$u^0 \geq u^1 \geq \dots \geq u^k \geq \dots \geq u_0.$$

Now we have

$$-\Delta u^{k+1} + M \frac{u^{k+1}}{d(x)^{1+\alpha}} \geq \frac{1}{(u^k)^\alpha} + M \frac{u^k}{d(x)^{1+\alpha}}$$

and

$$-\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^{1+\alpha}} = \frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^{1+\alpha}}.$$

It remains only to prove that $u_k \leq u^k$, we have $u_0 \leq u^0$, now assume that $u_{k-1} \leq u^k$ and prove that $u_k \leq u^{k+1}$. Subtract the preceding formal, multiplying by $v = (u_{k+1} - u^{k+1})^+ \in H_0^1(\Omega)$ and integrating by parts with last hypothesis and lemma (3.3), we find

$$\int_{\Omega} (\nabla(u_{k+1} - u^{k+1})^+)^2 dx + \frac{M}{d(x)^{1+\alpha}} (u_{k+1} - u^{k+1})^+ dx \leq 0$$

giving again the result. Forthere we obtain two monotones sequences satisfying;

$$u_0 \leq u_1 \leq \dots u_k \leq \dots \leq u^k \leq \dots \leq u^1 \leq u^0.$$

by exploiting the regularity for the linear problem (3.4) and the estimate in the lemma (3.3), we find

$$\|u_k\| \leq c \left\| \frac{1}{u_{k-1}^\alpha} + \frac{Mu_{k-1}}{d^{1+\alpha}} \right\|_{H^{-1}(\Omega)} \leq c.$$

Where the constant c is independent of k . Thus there exists a subsequences still denoted u_k and u^k such that : $u_k \rightharpoonup \underline{u}$ and $u^k \rightharpoonup \bar{u}$ weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$, (notice that the imbedding $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$ is compact), therefore, by the inverse Lebesgue theorem the subsequence u_k converges to $\underline{u}$ a.e. in Ω . Next we should pass to the limit in equation (3.4). The weak formulation is

$$\int_{\Omega} (\nabla u_k \nabla v + M \frac{u_k v}{d^{1+\alpha}}) dx = \int_{\Omega} (\frac{v}{u_{k-1}^\alpha} + M \frac{u_{k-1} v}{d^{1+\alpha}}) dx$$

for any $v \in H_0^1(\Omega)$ therefor $\int_{\Omega} \nabla u_{k+1} \nabla v dx \rightarrow \int_{\Omega} \nabla \underline{u} \nabla v dx$, for the first term in the right-hand side by the definition of K we obtain

$$\frac{v}{u_{k-1}^\alpha} = d \frac{v}{du_{k-1}^\alpha} \leq d^{1-\alpha} \frac{v}{d}$$

By Cauchy-Shwarz and Hardy's inequality

$$\int_{\Omega} d^{1-\alpha} \frac{v}{d} dx \leq \left(\int_{\Omega} \left| \frac{v}{d} \right|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} |d^{1-\alpha}|^2 dx \right)^{\frac{1}{2}} \leq c \int_{\Omega} |\nabla v|^2 dx < \infty.$$

With $u_k \rightarrow \underline{u}$ a.e. in Ω , we can apply the dominate convergence theorem, this gives that

$$\int_{\Omega} \frac{1}{u_k^\alpha} v \, dx \rightarrow \int_{\Omega} \frac{1}{\underline{u}^\alpha} v \, dx, \quad k \rightarrow \infty.$$

For the terms on the both sides, we have to prove that

$$\int_{\Omega} (M \frac{u_k v}{d^{1+\alpha}} - M \frac{\underline{u} v}{d^{1+\alpha}}) dx \rightarrow 0, \quad k \rightarrow \infty.$$

We have

$$\int_{\Omega} (M \frac{u_k v}{d^{1+\alpha}} - M \frac{\underline{u} v}{d^{1+\alpha}}) dx = \int_{\Omega} M \frac{(u_k - \underline{u}) v}{d^{1+\alpha}} dx = \int_{\Omega} M \frac{u_k - \underline{u}}{d^\alpha} \frac{v}{d} dx,$$

by using Couchy-Shwarz inequality we obtain

$$\int_{\Omega} M \frac{(u_k - \underline{u}) v}{d^\alpha} \frac{v}{d} dx \leq \left(M \int_{\Omega} \left| \frac{(u_k - \underline{u})^2}{d^{2\alpha}} \right| dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| \frac{v}{d} \right|^2 dx \right)^{\frac{1}{2}}$$

and by Hardy's inequality

$$\left(M \int_{\Omega} \left| \frac{(u_k - \underline{u})^2}{d^{2\alpha}} \right| dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| \frac{v}{d} \right|^2 dx \right)^{\frac{1}{2}} \leq \|u_k - \underline{u}\|_{L^2(\Omega, c)} \|v\|$$

Such that $2\alpha = \beta$ so by lemma 3.5 $u_k \rightarrow \underline{u}$ strongly in $L^2(\Omega, c)$, giving again the result.

3.1.1 Ordered Sub and Supersolutions

Proposition 3.6 *There exists a subsolution (resp a supersolution) of the problem (P') satisfying the conditions in Theorem (3.2).*

Proof As a subsolution, we set $u_0 = c\phi_1$ where $-\Delta\phi_1 = \lambda_1\phi_1$ in Ω , $\phi = 0$ on $\partial\Omega$, $\phi_1 > 0$, $c > 0$, we have

$$-\Delta u_0 - \frac{1}{u_0^\alpha} = c\lambda_1\phi_1 - \frac{1}{c^\alpha\phi_1^\alpha} = \frac{c^{1+\alpha}\lambda_1\phi_1^{1+\alpha} - 1}{c^\alpha\phi_1^\alpha} \leq 0$$

for $c > 0$ small. As a supersolution, we set $u^0 = C\psi$, where $\psi \in C^1(\overline{\Omega}) \cap C^2(\Omega)$ is the unique solution to the linear problem

$$-\Delta\psi = \frac{1}{d(x)^\alpha} \quad \text{in } \Omega, \quad \psi = 0 \quad \text{on } \partial\Omega$$

(see theorem 2.1). Then using the fact that $\psi \sim d(x)$ (see lemma 2.10), we obtain

$$-\Delta u^0 - \frac{1}{(u^0)^\alpha} = \frac{C}{d^\alpha} - \frac{1}{(C\psi)^\alpha} = \frac{C^{\alpha+1}\psi^\alpha - C\psi^\alpha}{(C\psi)^\alpha d^\alpha} \geq 0$$

for $C > 0$ large. Consequently the proof of (3.1) can be deduced from theorem (3.2).

3.2 Uniqueness of Solutions for $0 < \alpha < 1$

Theorem 3.7 *The problem (P') , has a unique weak solution u belongs to $H_0^1(\Omega)$.*

Proof we assume that u, v are two solutions to the problem (P') , First, we assume that $u \leq v$ in Ω . Multiplying (P') for u by v and by u for v and integrating by parts on Ω , we obtain

$$\int_{\Omega} \nabla u \nabla v - \nabla v \nabla u \, dx = \int_{\Omega} \frac{v}{u^\alpha} \, dx - \int_{\Omega} \frac{u}{v^\alpha} \, dx = 0.$$

Then

$$\int_{\Omega} \frac{v}{u^\alpha} \, dx - \int_{\Omega} \frac{u}{v^\alpha} \, dx = \int_{\Omega} \frac{v^{\alpha+1} - u^{\alpha+1}}{v^\alpha u^\alpha} \, dx = 0$$

Since $v \geq u, u \equiv v$. Notice that all the above integrals are meaningful. Indeed, since $u, v \in k$ we have ,e.g., that $\int_{\Omega} \frac{v}{u^\alpha} dx \leq c \int_{\Omega} d(x)^{1-\alpha} dx < \infty$. If now $u \not\leq v$ and $v \not\leq u$, we have $u_0 \leq u, v \leq u^0$. Then $\underline{u} \leq u, \underline{u} \leq v$ and it follows from above that $\underline{u} = u = v$.

3.3 Existence for The case $\alpha = 1$

We study now the same problem (P') but for $\alpha = 1$. If we try to apply the arguments in the preceding section, we will find some difficulties to the fact the embedding in the lemma 3.5 is not compact any more for $\beta = 2$, which is precisely the critical exponent arising for $\alpha \geq 1$. As above with the preceding section, Keeping the same definition of the weak solution, we will prove the following main existence theorem,

Theorem 3.8 *The problem (P') has a positive weak solution u satisfying,*

$$c_1 \left(\ln \left(\frac{k}{d(x)} \right) \right)^{\frac{1}{2}} \leq u(x) \leq c_2 \left(\ln \left(\frac{k}{d(x)} \right) \right)^{\frac{1}{2}}.$$

To prove the last theorem we replace the assumption on the sub and supersolution in the Theorem 3.2 by the following.

Theorem 3.9 *Assume that there exists a subsolution u_0 and a supersolution u^0 such that $u_0 \leq u^0$ and that there exist constants $c_1, c_2 > 0$ and $k > 0$ large enough satisfying:*

$$0 < c_1 d(x) \left(\ln\left(\frac{k}{d(x)}\right) \right)^{\frac{1}{2}} \leq u_0(x) \leq u^0(x) \leq c_2 \left(\ln\left(\frac{k}{d(x)}\right) \right)^{\frac{1}{2}} \quad \forall x \in \Omega. \quad (3.12)$$

Then, there exists a minimal solution $\underline{u}$ and a maximal solution $\bar{u}$ to the problem (P') such that

$$u_0 \leq \underline{u} \leq \bar{u} \leq u^0.$$

and we define this time the weight function $b(x) = \frac{1}{d(x)^2}$ and the sub set

$$K = [u_0, u^0] = \{u \in L^2(\Omega, b) : u_0 \leq u \leq u^0\}$$

Lemma 3.10 . *There exists a positive constant $M \succ 0$ such that the mapping $G : K \rightarrow H^{-1}(\Omega)$ defined by $G(w) = \frac{1}{w} + M \frac{w}{d(x)^2}$ is well defined, continuous and increasing.*

Proof. As above we prove that $G(w) \in H^{-1}(\Omega)$, $\forall w \in K$, the linearity is immediate, so we have to prove the continuity. Let $z \in H_0^1(\Omega)$, we obtain for the first term of $G(w)$,

$$\left| \left\langle \frac{1}{w}, z \right\rangle \right| = \left| \int_{\Omega} \frac{z}{w} dx \right| = \int_{\Omega} \left| \frac{z}{d} \right| \frac{d}{w} dx,$$

by using inequality of Cauchy-Schwarz and the definition of K we obtain

$$\int_{\Omega} \left| \frac{z}{d} \right| \frac{d}{w} dx \leq c \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \left\| \frac{1}{(\ln(\frac{k}{d}))^{\frac{1}{2}}} \right\|_{L^2(\Omega)}.$$

Now by using Hardy's inequality

$$c \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \left\| \frac{1}{(\ln(\frac{k}{d}))^{\frac{1}{2}}} \right\|_{L^2(\Omega)} \leq c' \|z\|$$

Since

$$\left\| \frac{1}{(\ln(\frac{k}{d}))^{\frac{1}{2}}} \right\|_{L^2(\Omega)} = \left(\int_{\Omega} \left| \frac{1}{(\ln(\frac{k}{d}))^{\frac{1}{2}}} \right|^2 dx \right)^{\frac{1}{2}} \leq \left(\int_{\Omega} \left| \frac{1}{(\ln(\frac{2d}{d}))} \right| dx \right)^{\frac{1}{2}}.$$

For the second term of $G(w)$, we have for any $z \in H_0^1(\Omega)$,

$$\left| \left\langle \frac{w}{d^2}, z \right\rangle \right| = \left| \int_{\Omega} \frac{wz}{d^2} dx \right| = \left| \int_{\Omega} \frac{w}{d} \frac{z}{d} dx \right|.$$

By Cauchy-Schwarz and Hardy's inequality

$$\left| \int_{\Omega} \frac{w}{d} \frac{z}{d} dx \right| \leq \left\| \frac{w}{d} \right\|_{L^2(\Omega)} \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \leq c \|\nabla z\|_{L^2(\Omega)}.$$

Noticing that

$$c = \left\| \frac{w}{d} \right\|_{L^2(\Omega)} = \left(\int_{\Omega} \left| \frac{w^2}{d^2} \right| dx \right)^{\frac{1}{2}} = \|w\|_{L^2(\Omega, b)}.$$

Next we prove the continuity of G . For the first term, reasoning as above

$$\left| \int_{\Omega} \left(\frac{1}{w_n} - \frac{1}{w} \right) z dx \right| = \left| \int_{\Omega} \frac{w - w_n}{w_n w} \left(\frac{z}{d} \right) d dx \right| \leq c'_n \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \leq c'_n \|z\| dx.$$

Such that

$$c'_n = \left\| \frac{d(w_n - w)}{w_n w} \right\|_{L^2(\Omega)}^2 = \int_{\Omega} \left| \frac{d^2(w_n - w)^2}{w_n^2 w^2} \right| dx$$

by the definition of K we obtain

$$c'_n \leq c \int_{\Omega} \left| \frac{(w_n - w)^2}{d^2 (\ln(\frac{k}{d}))^2} \right| dx \leq C \int_{\Omega} \left| \frac{(w_n - w)^2}{d^2} \right| dx = C \|w_n - w\|_{L^2(\Omega, b)}^2$$

giving the result. For the second term, $\forall z \in H_0^1(\Omega)$ we write

$$\left| \left\langle \frac{w_n - w}{d^2}, z \right\rangle \right| = \left| \int_{\Omega} \frac{w_n - w}{d} \frac{z}{d} dx \right|$$

By Cauchy-Schwarz and Hardy's inequality

$$\left| \left\langle \frac{w_n - w}{d^2}, z \right\rangle \right| \leq c \left\| \frac{w_n - w}{d} \right\|_{L^2(\Omega)} \|z\| dx = c \|w_n - w\|_{L^2(\Omega, b)}^2 \|z\|$$

giving again the result. Now prove that G is increasing, we work in the bounded interval $[u, v] \subset K$, by using the definition of K we obtain

$$G(u) - G(v) = \frac{1}{u} + M \frac{u}{d^2} - \frac{1}{v} - M \frac{v}{d^2} \leq 0.$$

(here θ denotes the intermediate point in the segment $[u, v]$) We need a second result is the following

Lemma 3.11 *For any $h \in H^{-1}(\Omega)$, there exists a unique solution $z \in H_0^1(\Omega)$, to the linear problem*

$$\begin{cases} -\Delta u + \frac{Mu}{d(x)^2} &= h \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.13)$$

Proof. Indeed we apply the Lax-Milgram theorem. The associated bilinear form is

$$a(u, v) = \int_{\Omega} \nabla u \nabla v dx + M \int_{\Omega} \frac{uv}{d(x)^2} dx.$$

Now we have to prove the continuity of $a(u, v)$,

$$|a(u, v)| \leq \int_{\Omega} |\nabla u \nabla v| dx + M \int_{\Omega} \left| \frac{u}{d} \right| \left| \frac{v}{d} \right| dx.$$

By using the inequality of Cauchy-Schwarz we obtain

$$|a(u, v)| \leq \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} + M \left\| \frac{u}{d} \right\|_{L^2(\Omega)} \left\| \frac{v}{d} \right\|_{L^2(\Omega)}.$$

We apply Hardy's inequality for the second term we find

$$|a(u, v)| \leq \|u\| \|v\| + M \|\nabla u\|_{L^2(\Omega)} \|\nabla v\|_{L^2(\Omega)} \leq (M+1) \|u\| \|v\|.$$

Now we prove that $a(u, v)$ is coercive

$$a(u, u) = \int_{\Omega} |\nabla u|^2 dx + \int_{\Omega} M \frac{u^2}{d^2} dx \geq \int_{\Omega} |\nabla u|^2 dx = \|u\|.$$

With $h \in H^{-1}(\Omega)$ so by Lax-Milgram theorem there exists unique solution $u \in H_0^1(\Omega)$ of (3.13).

Proof of Theorem 3.9 Problem (P') is obviously equivalent to the nonlinear problem

$$\begin{cases} -\Delta u + M \frac{u}{d(x)^2} &= \frac{1}{u} + M \frac{u}{d(x)^2} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.14)$$

by the lemma 3.11 and 3.10 we construct the following iterative scheme starting from the subsolution u_0 such that $u_{k+1} \in H_0^1(\Omega)$ to be the unique weak solution of the linear boundary value problem,

$$\begin{cases} -\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} &= \frac{1}{u_k} + M \frac{u_k}{d(x)^2} \\ u_{k+1} &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.15)$$

As above with the preceding section we prove that

$$u_0 \leq u_1 \leq \dots u_k \leq \dots \leq u^k \leq \dots \leq u^1 \leq u^0.$$

By the definition of subsolution u_0 , we have

$$\int_{\Omega} \nabla u_0 \nabla v dx - \int_{\Omega} (u_0^{-\alpha}) v dx \leq 0, \forall v > 0 \quad (3.16)$$

First we confirm this for $k = 0$ in (3.15), we have

$$\begin{cases} -\Delta u_1 + M \frac{u_1}{d(x)^2} &= \frac{1}{u_0} + M \frac{u_0}{d(x)^2} \text{ in } \Omega \\ u_1 &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.17)$$

subtract (3.17) from (3.16), multiplying by $v = (u_0 - u_1)^+ \in H_0^1(\Omega)$, $v \geq 0$ and using integration by parts we find

$$\int_{\Omega} \nabla (u_0 - u_1)^+ dx + \int_{\Omega} M \frac{u_1}{d(x)^2} (u_0 - u_1)^+ dx \leq 0 \quad (3.18)$$

so

$$\nabla (u_0 - u_1)^+ dx = 0 \implies u_0 \leq u_1.$$

Now we have to prove that $u_k \leq u_{k+1}$, for this we assume that,

$$u_{k-1} \leq u_k. \quad (3.19)$$

By multiplying (3.15) by $v \in H_0^1(\Omega)$ and integrating by parts, we find

$$\int_{\Omega} \nabla u_{k+1} \nabla v + M \frac{u_{k+1}}{d(x)^2} v dx = \int_{\Omega} \left(\frac{1}{u_k} + M \frac{u_k}{d(x)^2} \right) v dx$$

and

$$\int_{\Omega} \nabla u_k \nabla v + M \frac{u_k}{d(x)^2} v dx = \int_{\Omega} \left(\frac{1}{u_{k-1}} + M \frac{u_{k-1}}{d(x)^2} \right) v dx.$$

Subtract the preceding formulas, substituting v by $(u_k - u_{k-1})^+$, we obtain

$$\int_{\Omega} \left((\nabla (u_k - u_{k+1})^+)^2 + \frac{M u_1}{d(x)^2} (u_k - u_{k+1})^+ \right) dx = \int_{\Omega} (G(u_{k-1}) - G(u_k)) (u_k - u_{k+1})^+ dx$$

by lemma 3.10 and 3.19

$$G(u_{k-1}) - G(u_k) \leq 0 \implies u_k \leq u_{k+1}$$

Next we assume that $u_k \leq u^0$ and prove that $u_{k+1} \leq u^0$, we have

$$-\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} = \frac{1}{u_k} + M \frac{u_k}{d(x)^2}$$

and

$$-\Delta u^0 + M \frac{u^0}{d(x)^2} \geq \frac{1}{u^0} + M \frac{u^0}{d(x)^2}$$

subtract the preceding formulas, multiplying by $v = (u_{k+1} - u^0)^+ \in H_0^1(\Omega)$, using integration by parts and by lemma 3.10 we obtain

$$\begin{aligned} \int_{\Omega} \left((\nabla(u_{k+1} - u^0) + \frac{M}{d(x)^2}(u_{k+1} - u^0)^+) \right) dx &\leq \int_{\Omega} (G(u_k) - G(u^0))(u_{k+1} - u^0)^+ dx \\ &\leq 0. \end{aligned}$$

So

$$u_{k+1} \leq u^0$$

Thus we have established the increasing sequence;

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq u^0.$$

By the same way we establish the second decreasing sequence;

$$u^0 \geq u^1 \geq \dots \geq u^k \geq \dots \geq u_0.$$

It remains only to prove that $u_k \leq u^k$, we have $u_0 \leq u^0$, now we assume that $u_{k-1} \leq u^k$ and we prove that $u_k \leq u^{k+1}$, we have

$$-\Delta u^{k+1} + M \frac{u^{k+1}}{d(x)^2} \geq \frac{1}{u^k} + M \frac{u^k}{d(x)^2}.$$

and

$$-\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} = \frac{1}{u_k} + M \frac{u_k}{d(x)^2}.$$

Subtract the preceding formulas, multiplying by $v = (u_{k+1} - u^{k+1})^+ \in H_0^1(\Omega)$, using integration by parts with last hypothesis and lemma (3.10), we find

$$\int_{\Omega} (\nabla(u_{k+1} - u^{k+1})^+)^2 dx + \frac{M}{d(x)^2} (u_{k+1} - u^{k+1})^+ dx \leq 0,$$

giving again the result. as above we obtain two monotone sequences satisfying;

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq u^k \dots \leq u^1 \leq u^0.$$

By exploiting the regularity for the problem in lemma 3.11 and the estimates in lemma 3.10, we obtain

$$\|u_k\| dx \leq c \left\| \frac{1}{u_{k-1}} + \frac{Mu_{k-1}}{d^2} \right\|_{H^{-1}(\Omega)} \leq c.$$

Where c is a constant independent of k . So there exists a subsequence still do note u_k by the inverse Lebesgue theorem $u_k \rightarrow \underline{u}$ (by the same way for the second subsequence $u^k \rightarrow \bar{u}$) weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Next we should pass to the limit in equation (3.15), the weak formulation is

$$\int_{\Omega} \nabla u_k \nabla v + M \frac{u_k v}{d(x)^2} dx = \int_{\Omega} \frac{v}{u_{k-1}} + M \frac{u_{k-1} v}{d(x)^2} dx$$

for any $v \in H_0^1(\Omega)$, $\int_{\Omega} \nabla u_k \nabla v dx \rightarrow \int_{\Omega} \nabla \underline{u} \nabla v dx$ clearly. Concerning the first term on right-hand side, by Cauchy-Schwarz and Hardy's inequality and the fact that $u \in K$ we obtain

$$\frac{v}{u_{k-1}} = \frac{v}{d} \frac{d}{u_{k-1}} \leq c \frac{v}{d} \frac{1}{\left(\ln\left(\frac{k}{d}\right)\right)^{\frac{1}{2}}} \leq C \|v\| < \infty.$$

Such that

$$C = \left(\int_{\Omega} \frac{1}{\left(\ln\left(\frac{k}{d}\right)\right)^{\frac{1}{2}}} dx \right)^{\frac{1}{2}} < \infty$$

and we have $u_k \rightarrow \underline{u}$ a.e. in Ω . So we can apply the dominated convergence theorem this gives that

$$\int_{\Omega} \frac{1}{u_k} v \rightarrow \int_{\Omega} \frac{1}{\underline{u}} v dx, \quad k \rightarrow \infty.$$

For the second terms on both sides we have to prove that

$$\int_{\Omega} M \frac{u_{k-1} v}{d(x)^2} dx \rightarrow \int_{\Omega} M \frac{\underline{u} v}{d(x)^2} dx, \quad k \rightarrow \infty.$$

By the definition of K we have

$$M \frac{u_{k-1} v}{d(x)^2} = M \frac{v}{d} \frac{u_{k-1}}{d} \leq \frac{v}{d} \frac{1}{\left(\ln\left(\frac{k}{d}\right)\right)^{\frac{1}{2}}}$$

By Cauchy-Schwarz and Hardy's inequality we find

$$\left\| \frac{v}{d \left(\ln\left(\frac{k}{d}\right) \right)^{\frac{1}{2}}} \right\|_{L^1(\Omega)} \leq c \|v\| \left\| \frac{1}{\left(\ln\left(\frac{k}{d}\right) \right)^{\frac{1}{2}}} \right\|_{L^2(\Omega)} < \infty.$$

As above using the dominated convergence theorem is giving again the result.

3.3.1 Ordered Sub and Supersolutions

Theorem 3.12 *There exist a subsolution u_0 (resp a supersolution u^0) of problem (P') satisfying the conditions in theorem (3.2).*

Proof. As a subsolution, we set $u_0 = c\phi_1 \left(\ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right) \right)$ where

$$-\Delta \left(c\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right) \right) = \frac{1}{c\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right)} \quad \text{in } \Omega.$$

Such that $\phi_1 = 0$ on $\partial\Omega$, $\phi_1 > 0$, $c > 0$, We have

$$-\Delta u_0 - \frac{1}{u_0} = c\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right) - \frac{1}{c\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right)} = \frac{c^2\phi_1^2 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right) - 1}{c\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right)} \leq 0$$

For $c > 0$ small. As a supersolution, we set

$$u^0 = C\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right)$$

where

$$-\Delta \left(C\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right) \right) = \frac{1}{C\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right)} \quad \text{in } \Omega.$$

Such that $\phi_1 = 0$ on $\partial\Omega$, $\phi_1 > 0$, $C > 0$.

$$-\Delta u^0 - \frac{1}{u^0} = C\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right) - \frac{1}{C\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right)} = \frac{C^2\phi_1^2 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right) - 1}{C\phi_1 \ln^{\frac{1}{2}}\left(\frac{k}{\phi_1}\right)} \geq 0$$

For $C > 0$ large. So we can easily conclude the theorem (3.9). which gives the theorem (3.8) clearly.

3.4 Uniqueness for $\alpha = 1$

Theorem 3.13 *The problem (P') has a unique weak positive solution $u \in H_0^1(\Omega)$.*

Proof Let u, v are two solutions to the problem (P') . First, we assume that $u \leq v$ in Ω . Multiplying (P') for u by v and by u for v , integrating by parts we obtain

$$\int_{\Omega} \nabla u \nabla v - \nabla v \nabla u \, dx = \int_{\Omega} \frac{v}{u} \, dx - \int_{\Omega} \frac{u}{v} \, dx = 0.$$

Then

$$\int_{\Omega} \frac{v}{u} \, dx - \int_{\Omega} \frac{u}{v} \, dx = \int_{\Omega} \frac{v^2 - u^2}{vu} \, dx = 0$$

Since $v \geq u$, $u \equiv v$. Notice that all the above integrals are meaningful.

Indeed, since $u, v \in k$ we have ,e.g., that

$$\int_{\Omega} \frac{v}{u} \, dx \leq c \int_{\Omega} \frac{(\ln(\frac{k}{d}))^{\frac{1}{2}}}{(\ln(\frac{k}{d}))^{\frac{1}{2}}} \, dx = c < \infty.$$

If now $u \not\leq v$ and $v \not\leq u$, we have $u_0 \leq u, v \leq u^0$. Then $\underline{u} \leq u, \underline{u} \leq v$ and it follows from above that $\underline{u} = u = v$.

3.5 Existence for The case $1 < \alpha < 3$

We study now the same problem (P') but for $1 < \alpha < 3$. As above we have to prove the following main existence theorem.

Theorem 3.14 *The problem (P') has a positive weak solution u satisfying,*

$$c_1 d(x)^{\frac{2}{1+\alpha}} \leq u(x) \leq c_2 d(x)^{\frac{2}{1+\alpha}}.$$

Now we replace the assumption on the sub and supersolution in the Theorem 3.2 by the following.

Theorem 3.15 *Assume that there exists a subsolution u_0 and a supersolution u^0 such that $u_0 \leq u^0$ and that there exist constants c_1, c_2 satisfying:*

$$0 < c_1 d(x)^{\frac{2}{1+\alpha}} \leq u_0(x) \leq u^0(x) \leq c_2 d(x)^{\frac{2}{1+\alpha}} \quad \text{in } \Omega. \quad (3.20)$$

Then, there exists a minimal solution $\underline{u}$ and a maximal solution $\bar{u}$ such that

$$u_0 \leq \underline{u} \leq \bar{u} \leq u^0.$$

We define now the weight function $b(x) = \frac{1}{d(x)^2}$ and the sub set

$$K = [u_0, u^0] = \{u \in L^2(\Omega, b) : u_0 \leq u \leq u^0\}.$$

Notice that K is convex, closed and bounded. A first auxiliary result is the following.

Lemma 3.16 . *There exists a positive constant $M > 0$ such that the mapping $G : K \rightarrow H^{-1}(\Omega)$ defined by $G(w) = \frac{1}{w^\alpha} + M \frac{w}{d(x)^2}$ is well defined, continuous and increasing.*

Proof. First we prove that G is well defined, the linearity is immediate, so we have to prove the continuity. Let $z \in H_0^1(\Omega)$ we obtain for the first term of $G(w)$

$$\left| \left\langle \frac{1}{w^\alpha}, z \right\rangle \right| = \left| \int_{\Omega} \frac{z}{w^\alpha} dx \right| = \int_{\Omega} \left| \frac{z}{d} \right| \frac{d}{w^\alpha} dx$$

by using inequality of Cauchy Schwarz and the definition of K we obtain

$$\int_{\Omega} \left| \frac{z}{d} \right| \frac{d}{w^\alpha} dx \leq c \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \left\| d^{1-\frac{2\alpha}{1+\alpha}} \right\|_{L^2(\Omega)}.$$

Now by using Hardy's inequality

$$c \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \left\| d^{1-\frac{2\alpha}{1+\alpha}} \right\|_{L^2(\Omega)} \leq c' \|z\|$$

Since

$$\left\| d^{1-\frac{2\alpha}{1+\alpha}} \right\|_{L^2(\Omega)} = \int_{\Omega} d^{\frac{2(1-\alpha)}{1+\alpha}} dx < \infty.$$

(we have in fact $\frac{2(1-\alpha)}{1-\alpha} + 1 = \frac{3-\alpha}{1+\alpha} > 0$)

For the second term of $G(w)$ is the same with the preceding section. Next we prove the continuity of G . For the first term, reasoning as above

$$\left| \int_{\Omega} \left(\frac{1}{w_n^\alpha} - \frac{1}{w^\alpha} \right) z dx \right| = \left| \int_{\Omega} \frac{w^\alpha - w_n^\alpha}{w^\alpha w_n^\alpha} \left(\frac{z}{d} \right) d dx \right| \leq c'_n \left\| \frac{z}{d} \right\|_{L^2(\Omega)} \leq c'_n \|z\| dx$$

. Such that

$$c'_n = \left\| \frac{d(w_n^\alpha - w^\alpha)}{w_n^\alpha w^\alpha} \right\|_{L^2(\Omega)}^2$$

And using as above the mean value theorem we find

$$c'_n = \left\| \frac{d(w_n - w) \alpha w(\theta)^{\alpha-1}}{w_n^\alpha w^\alpha} \right\|_{L^2(\Omega)}^2 = \int_{\Omega} \left| \frac{d^2(w_n - w)^2 \alpha^2 w(\theta)^{2(\alpha-1)}}{w_n^2 \alpha w^2} \right| dx$$

and by the definition of K

$$\begin{aligned} c'_n &\leq c \int_{\Omega} \frac{|w_n - w|^2 d^{\frac{4(\alpha-1)}{\alpha+1}} \alpha^2}{d^{\frac{8\alpha}{\alpha+1}}} dx \\ c'_n &\leq c' \int_{\Omega} \frac{|w_n - w|^2 d^2}{d^4} dx \\ c'_n &\leq c' \|w_n - w\|_{L^2(\Omega, b)}^2 \end{aligned}$$

giving the result. For the second term is above with preceding section. On the other side, we prove that G is increasing, we work in the bounded interval $[u, v] \subset K$, by using the definition of K , the mean value theorem and we choose M large, we obtain

$$\begin{aligned} G(u) - G(v) &= \frac{1}{u^\alpha} + M \frac{u}{d^2} - \frac{1}{v^\alpha} - M \frac{v}{d^2} \\ &= -\alpha w(\theta)^{-\alpha-1} (u - v) - \frac{2M}{d^2} (u - v) \\ &\leq (M - \alpha) \frac{u-v}{d^{1+\alpha}} \leq 0 \end{aligned} \tag{3.21}$$

(here θ denotes the intermediate point in the segment $[u, v]$).

Proof of Theorem 3.15 Problem (P') is obviously equivalent to the nonlinear problem

$$\begin{cases} -\Delta u + M \frac{u}{d(x)^2} = \frac{1}{u^\alpha} + M \frac{u}{d(x)^2} & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{3.22}$$

By the lemma 3.11 and 3.16 we construct the following iterative scheme starting from the subsolution u_0 such that $u_{k+1} \in H_0^1(\Omega)$ to be the unique weak solution of the linear boundary value problem.

$$\begin{cases} \Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} = \frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^2} \\ u_{k+1} = 0 & \text{on } \partial\Omega \end{cases} \tag{3.23}$$

We have to prove that

$$u_0 \leq u_1 \leq \dots u_k \leq \dots \leq u^k \leq \dots \leq u^1 \leq u^0.$$

we have by the definition of subsolution u_0 ,

$$\int_{\Omega} \nabla u_0 \nabla v dx - \int_{\Omega} (u_0^{-\alpha}) v dx \leq 0. \tag{3.24}$$

First we prove that $u_0 \leq u_1$, for this we put $k = 0$ in (3.23), we have

$$\begin{cases} \Delta u_1 + M \frac{u_1}{d(x)^2} &= \frac{1}{u_0^\alpha} + M \frac{u_0}{d(x)^2} \text{ in } \Omega \\ u_1 &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.25)$$

Subtract (3.25) from (3.1), multiplying by $v = (u_0 - u_1)^+ \in H_0^1(\Omega)$, $v \geq 0$ and using integration by parts we obtain

$$\int_{\Omega} \nabla(u_0 - u_1)^{+2} dx + \int_{\Omega} M \frac{u_1}{d(x)^2} (u_0 - u_1)^{+2} dx \leq 0 \quad (3.26)$$

so $\int_{\Omega} \nabla(u_0 - u_1)^{+2} dx = 0$ which implies $(u_0 - u_1)^+ = 0$ and this gives $u_0 \leq u_1$.

Now we have to prove that $u_k \leq u_{k+1}$, for this we assume that

$$u_{k-1} \leq u_k \quad (3.27)$$

From (3.23) for each $v \in H_0^1(\Omega)$ we find

$$\int_{\Omega} \nabla u_{k+1} \nabla v + M \frac{u_{k+1}}{d(x)^2} v dx = \int_{\Omega} \left(\frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^2} \right) v dx.$$

and

$$\int_{\Omega} \nabla u_k \nabla v + M \frac{u_k}{d(x)^2} v dx = \int_{\Omega} \left(\frac{1}{u_{k-1}^\alpha} + M \frac{u_{k-1}}{d(x)^2} \right) v dx.$$

Subtract the preceding formulas, substituting v by $(u_k - u_{k+1})^+$, we obtain

$$\int_{\Omega} \left((\nabla(u_k - u_{k+1}))^{+2} + \frac{M u_1}{d(x)^2} (u_k - u_{k+1})^{+2} \right) dx = \int_{\Omega} (G(u_{k-1}) - G(u_k)) (u_k - u_{k+1})^+ dx,$$

by lemma 3.16 and 3.27

$$G(u_{k-1}) - G(u_k) \leq 0 \implies u_k \leq u_{k+1}.$$

Next we assume that $u_k \leq u^0$ and prove that $u_{k+1} \leq u^0$, we have

$$-\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} = \frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^2}$$

and

$$-\Delta u^0 + M \frac{u^0}{d(x)^2} \geq \frac{1}{(u^0)^\alpha} + M \frac{u^0}{d(x)^2}.$$

Subtract the preceding formulas, multiplying by $v = (u_{k+1} - u^0)^+ \in H_0^1(\Omega)$, using integration by parts and lemma 3.16, we find

$$\int_{\Omega} (\nabla(u_{k+1} - u^0) + \frac{M}{d(x)^2} (u_{k+1} - u^0)^{+2}) dx \leq 0.$$

So $u_{k+1} \leq u^0$. Thus we have established the increasing sequence;

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq u^0.$$

By the same way we establish the second decreasing sequence;

$$u^0 \geq u^1 \geq \dots \geq u^k \geq \dots \geq u_0.$$

It remains only to prove that $u_k \leq u^k$, we have $u_0 \leq u^0$, assume that $u_{k-1} \leq u^{k-1}$ and prove that $u_{k+1} \leq u^{k+1}$, we have

$$-\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} = \frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^2}$$

and

$$-\Delta u^{k+1} + M \frac{u^{k+1}}{d^2} \geq \frac{1}{u^{(k)\alpha}} + M \frac{u^k}{d^2}.$$

Subtract the preceding formulas, multiplying by $v = (u_{k+1} - u^{k+1})^+ \in H_0^1(\Omega)$, using integration by parts, by the last hypothesis and lemma 3.16 we find

$$\int_{\Omega} (\nabla(u_{k+1} - u^{k+1})^+)^2 dx + \frac{M}{d^2} (u_{k+1} - u^{k+1})^+ dx \leq 0$$

giving again the result. From this we obtain two monotone sequences satisfying;

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq u^k \dots \leq u^1 \leq u^0.$$

By the exploiting the regularity for the problem in lemma 3.11 and the estimates in lemma 3.16 we obtain

$$\|u_k\| \leq c \left\| \frac{u_{k-1}^\alpha}{d} + \frac{Mu_{k-1}}{d^2} \right\|_{H^{-1}(\Omega)} \leq c.$$

Where c is a constant independent of k . So there exists subsequence still do notes u_k and by the inverse Lebesgue theorem $u_k \rightharpoonup \underline{u}$ (by the same way for the second subsequence $u^k \rightharpoonup \bar{u}$) weakly in $H_0^1(\Omega)$ and strongly in $L^2(\Omega)$. Next we should pass to the limit in equation (3.23) the weak formulation is

$$\int_{\Omega} \nabla u_k \nabla v + M \frac{u_k v}{d(x)^2} dx = \int_{\Omega} \frac{v}{u_{k-1}^\alpha} + M \frac{u_{k-1} v}{d(x)^2} dx$$

for any $v \in H_0^1(\Omega)$ therefor $\int_{\Omega} \nabla u_k \nabla v dx \rightharpoonup \int_{\Omega} \nabla \underline{u} \nabla v dx$. Concerning the first term on right-hand side we have by the definition of K

$$\frac{v}{u_{k-1}^\alpha} = \frac{v}{d} \frac{d}{u_{k-1}^\alpha} \leq c \frac{v}{d} d^{\frac{1-\alpha}{1+\alpha}}$$

by Cauchy-Schwarz and Hardy's inequality

$$\int_{\Omega} \frac{v}{d} d^{\frac{1-\alpha}{1+\alpha}} dx \leq C \|v\| < \infty.$$

Such that

$$C = \left(\int_{\Omega} d^{\frac{2(1-\alpha)}{1+\alpha}} dx \right)^{\frac{1}{2}} < \infty$$

and we have $u_k \rightarrow \underline{u}$ a.e. in Ω . So we can apply the dominated convergence theorem this gives that

$$\int_{\Omega} \frac{1}{u_k^{\alpha}} v \rightarrow \int_{\Omega} \frac{1}{\underline{u}^{\alpha}} v dx, \quad k \rightarrow \infty.$$

For the second terms on both sides we have to prove that

$$\int_{\Omega} M \frac{u_{k-1} v}{d(x)^2} dx \rightarrow \int_{\Omega} M \frac{\underline{u} v}{d(x)^2} dx, \quad k \rightarrow \infty.$$

We have

$$M \frac{u_{k-1} v}{d(x)^2} = M \frac{v}{d} \frac{u_{k-1}}{d}.$$

By the definition of K we obtain

$$\left| \frac{u_{k-1}}{d} \right| \leq c d^{\frac{1-\alpha}{1+\alpha}}$$

By Cauchy-Schwarz and Hardy's inequality

$$c \int_{\Omega} \frac{v}{d} d^{\frac{1-\alpha}{1+\alpha}} dx \leq c \left(\int_{\Omega} |\nabla v|^2 dx \right)^{\frac{1}{2}} \left(\int_{\Omega} \left| d^{\frac{1-\alpha}{1+\alpha}} \right|^2 dx \right)^{\frac{1}{2}} < \infty$$

Using as above the dominated convergence theorem is giving again the result.

3.5.1 Ordered Sub and Supersolutions

Proposition 3.17 *There exist a subsolution u_0 (resp a supersolution u^0) to the problem (P') satisfying the conditions in theorem 3.15.*

Proof. As a subsolution, we set $u_0 = c\phi_1^t$ with $t = \frac{2}{1+\alpha} < 1$, where

$$-\Delta \phi_1^t = \phi_1^{t-2} (t(1-t) |\nabla \phi_1|^2 + \lambda_1 t \phi_1^2) = \lambda_1 t \phi_1^t + t(1-t) \phi_1^{t-2} |\nabla \phi_1|^2$$

Hence we obtain

$$-\Delta u_0 - \frac{1}{u_0} = ct(1-t) \phi_1^{t-2} |\phi_1|^2 + c\lambda_1 t \phi_1^t - \frac{1}{c^{\alpha} \phi_1^t}$$

$$\frac{ct(t-1)|\nabla\phi_1|^2}{\phi_1^{\frac{2\alpha}{1+\alpha}}} + \lambda_1 ct\phi_1^t - \frac{1}{c^\alpha}\phi_1^{\frac{2\alpha}{1+\alpha}} \leq 0$$

using that $t-2 = \frac{-2\alpha}{1+\alpha}$, and this is equivalent to

$$t(1-t) = |\nabla\phi_1|^2 + \lambda_1 t \leq \frac{1}{c^{\alpha+1}}$$

which is satisfied for $c > 0$ small.

Reasoning in a similar way for the supersolution $u^0 = C\phi_1^t$, we infer that

$$t(1-t)|\nabla\phi_1|^2 + \lambda_1 t\phi_1^{t+\frac{2}{\alpha+1}} \geq \frac{1}{c^{\alpha+1}}$$

We know that $|\nabla\phi_1| \geq \delta_1 > 0$ in $\Omega_\xi = \{x \in \Omega \mid d(x) \leq \xi\}$ for some $\delta_2 > 0$. Then

$$t(1-t)|\nabla\phi_1|^2 \geq t(1-t)\delta_1^2 \geq \frac{1}{c^{\alpha+1}},$$

on Ω_ξ $C > C_1 > 0$ large enough. On Ω/Ω_ξ , we have that $\phi_1 \geq \delta_2$ some δ_2 and it is enough

to have

$$\lambda_1 t\phi_1^{t+\frac{2}{\alpha+1}} \geq \frac{1}{c^{\alpha+1}}$$

which is satisfied for $C > C_2$ for some $C_2 > 0$ large enough. Finally we pick

$C > \max(C_1, C_2)$. So we can easily conclude the theorem (3.15). Which gives the theorem (3.14) clear.

3.6 Uniqueness for $1 < \alpha < 3$

Theorem 3.18 *The problem (P') has a unique positive weak solution $u \in H_0^1(\Omega)$.*

Proof. Let u, v two solutions to the problem (P') . First, we assume that $u \leq v$ in Ω .

Multiplying (P') for u by v and by u for v and integrating by parts on Ω , we obtain

$$\int_{\Omega} \nabla u \nabla v - \nabla v \nabla u \, dx = \int_{\Omega} \frac{v}{u^\alpha} \, dx - \int_{\Omega} \frac{u}{v^\alpha} \, dx = 0.$$

Then

$$\int_{\Omega} \frac{v}{u^\alpha} \, dx - \int_{\Omega} \frac{u}{v^\alpha} \, dx = \int_{\Omega} \frac{v^{\alpha+1} - u^{\alpha+1}}{v^\alpha u^\alpha} \, dx = 0$$

Since $v \geq u, u \equiv v$. Notice that all the above integrals are meaningful. Indeed, since $u, v \in K$ we have ,e.g., that

$$\int_{\Omega} \frac{v}{u^{\alpha}} dx \leq c \int_{\Omega} d(x)^{\frac{2(1-\alpha)}{1+\alpha}} dx < \infty.$$

Such that $(\frac{2(1-\alpha)}{1+\alpha} + 1 > 0)$. If now $u \not\leq v$ and $v \not\leq u$, we have $u_0 \leq u, v \leq u^0$. Then $\underline{u} \leq u, \underline{u} \leq v$ and it follows from above that $\underline{u} = u = v$.

3.7 Existence for The case $\alpha \geq 3$

In this section we will study the same problem (P') for $\alpha \geq 3$. The weak solution will not be in $W_0^{1,p}(\Omega)$, so we need the following definition

Definition 3.5 *A weak solution of problem (P') is a function $u \in W_0^{1,p}(\Omega)$ for some $p > 1$, satisfies*

$$\int_{\Omega} \nabla u \nabla v \, dx = \int_{\Omega} f v \, dx,$$

$\forall v \in W_0^{1,p'}(\Omega)$.

So, we prove the following theorem

Theorem 3.19 *The problem (P') has a positive weak solution u satisfying,*

$$c_1 d(x)^{\frac{2}{1+\alpha}} \leq u(x) \leq c_2 d(x)^{\frac{2}{1+\alpha}}.$$

As above, we need to prove the following main existence theorem.

Theorem 3.20 *we assume that there exists a subsolution u_0 and supersolution u^0 such that $u_0 \leq u^0$ and that there exist constants c_1, c_2 satisfying:*

$$0 < c_1 d(x)^{\frac{2}{1+\alpha}} \leq u_0(x) \leq u^0(x) \leq c_2 d(x)^{\frac{2}{1+\alpha}} \quad \forall x \in \Omega. \quad (3.28)$$

Then, there exists a minimal solution $\underline{u}$ and maximal solution $\bar{u}$ such that

$$u_0 \leq \underline{u} \leq \bar{u} \leq u^0.$$

As above we define the weight function $b(x) = \frac{1}{d(x)^2}$ and the subset

$$K = [u_0, u^0] = \{u \in L^p(\Omega, b) : u_0 \leq u \leq u^0\}.$$

Then in order to prove the theorem 3.20, we need the following lemma.

Lemma 3.21 *There exists a positive constant $M > 0$ such that the mapping $G : K \rightarrow W^{-1,p}(\Omega)$ defined by $G(w) = \frac{1}{w^\alpha} + M \frac{w}{d(x)^2}$ is well defined, continuous and increasing.*

Proof. First we prove that $G(w) \in W^{-1,p}(\Omega)$, $\forall w \in K$, the linearity is immediate, so we have to prove the continuity.

Let $z \in W_0^{1,p'}(\Omega)$ we obtain for the first term of $G(w)$

$$\left| \left\langle \frac{1}{w^\alpha}, z \right\rangle \right| = \left| \int_\Omega \frac{z}{w^\alpha} dx \right| \leq \int_\Omega \left| \frac{z}{d} \right| \frac{d}{w^\alpha} dx,$$

by using inequality of Holder and the definition of K we obtain

$$\int_\Omega \left| \frac{z}{d} \right| \frac{d}{w^\alpha} dx \leq \left(c \int_\Omega \left| \frac{z}{d} \right|^{p'} dx \right)^{\frac{1}{p'}} \left(\int_\Omega \left| d^{1-\frac{2\alpha}{1+\alpha}} \right|^p dx \right)^{\frac{1}{p}}.$$

Now by using Hardy's inequality

$$\int_\Omega \left| \frac{z}{d} \right| \frac{d}{w^\alpha} dx \leq \left(c \int_\Omega |\nabla z|^{p'} dx \right)^{\frac{1}{p'}} \left(\int_\Omega \left| d^{1-\frac{2\alpha}{1+\alpha}} \right|^p dx \right)^{\frac{1}{p}} \leq c' \|z\|_{W_0^{1,p'}(\Omega)},$$

since

$$c' = \left(\int_\Omega \left| d^{1-\frac{2\alpha}{1+\alpha}} \right|^p dx \right)^{\frac{1}{p}} < \infty.$$

(We have in fact $(1 - \frac{2\alpha}{1+\alpha})p > -1$)

For the second term of $F(w)$ we have, for all $z \in W_0^{1,p'}(\Omega)$

$$\left| \left\langle \frac{w}{d^2}, z \right\rangle \right| = \left| \int_\Omega \frac{wz}{d^2} dx \right| = \left| \int_\Omega \frac{w}{d} \frac{z}{d} dx \right|$$

By Holder and Hardy's inequality

$$\left| \int_\Omega \frac{w}{d} \frac{z}{d} dx \right| \leq \left(\int_\Omega \left| \frac{w}{d} \right|^p dx \right)^{\frac{1}{p}} \left(\int_\Omega \left| \frac{z}{d} \right|^{p'} dx \right)^{\frac{1}{p'}} \leq c \|z\|_{W_0^{1,p'}(\Omega)}.$$

Noticing that

$$c = \left(\int_\Omega \left| \frac{w}{d} \right|^p dx \right)^{\frac{1}{p}} = \left(\int_\Omega \frac{|w|^p}{d^p} \frac{d^{2+p}}{d^{2+p}} dx \right)^{\frac{1}{p}} = \left(\int_\Omega \frac{|w|^p}{d^2} d^{2-p} dx \right)^{\frac{1}{p}} = C \|w\|_{L^p(\Omega, b)}.$$

Next, we prove the continuity of G . For the first term, reasoning as above

$$\left| \int_{\Omega} \left(\frac{1}{w_n^{\alpha}} - \frac{1}{w^{\alpha}} \right) z dx \right| = \left| \int_{\Omega} \frac{w^{\alpha} - w_n^{\alpha}}{w^{\alpha} w_n^{\alpha}} \left(\frac{z}{d} \right) d dx \right| \leq c_n \left\| \frac{z}{d} \right\|_{L^p(\Omega)} \leq c_n \|z\|_{W_0^{1,p}(\Omega)}.$$

Such that

$$c_n = \left\| \frac{d(w_n^{\alpha} - w^{\alpha})}{w_n^{\alpha} w^{\alpha}} \right\|_{L^p(\Omega)}^p,$$

and using as above the mean value theorem, we find

$$c'_n = \left\| \frac{d(w_n - w) \alpha w(\theta)^{\alpha-1}}{w_n^{\alpha} w^{\alpha}} \right\|_{L^p(\Omega)}^p = \int_{\Omega} \left| \frac{d^p (w_n - w)^p \alpha^p w(\theta)^{p(\alpha-1)}}{w_n^{p\alpha} w^{p\alpha}} \right| dx$$

and by the definition of K

$$c'_n \leq c \int_{\Omega} \frac{|w_n - w|^p d^{\frac{3\alpha p - p}{\alpha+1}} \alpha^p}{d^{\frac{4\alpha p}{\alpha+1}}} dx,$$

thus

$$c'_n \leq c' \int_{\Omega} |w_n - w|^p d^{-\frac{p(\alpha+1)}{\alpha+1}} dx = c' \int_{\Omega} |w_n - w|^p d^{-p} d^{\frac{p-2}{p-2}} dx = C \|w_n - w\|_{L^p(\Omega,b)}^p$$

giving the result. For the second term, any $z \in W_0^{1,p'}(\Omega)$ we write

$$\left| \left\langle \frac{w_n - w}{d^2}, z \right\rangle \right| = \left| \int_{\Omega} \frac{w_n - w}{d} \frac{z}{d} dx \right|$$

By Holder and Hardy's inequality

$$\left| \left\langle \frac{w_n - w}{d^2}, z \right\rangle \right| \leq c \left\| \frac{w_n - w}{d} \right\|_{L^p(\Omega)} \|z\|_{W_0^{1,p'}(\Omega)} = c \|w_n - w\|_{L^p(\Omega,b)} \|z\|_{W_0^{1,p'}(\Omega)}$$

where

$$\left\| \frac{w_n - w}{d} \right\|_{L^p(\Omega)} = \int_{\Omega} \frac{|w_n - w|^p}{d^2} d^{2-p} dx = c \|w_n - w\|_{L^p(\Omega,b)},$$

giving again the result. Now we prove that G is increasing, we work in

the bounded interval $[u, v] \subset K$ by using the definition of K , the mean

value theorem and choose the constant M large, we obtain

$$\begin{aligned} G(u) - G(v) &= \frac{1}{u^{\alpha}} + M \frac{u}{d^2} - \frac{1}{v^{\alpha}} - M \frac{v}{d^2} \\ &= -\alpha w(\theta)^{-\alpha-1} (u - v) - \frac{M}{d^2} (u - v) \end{aligned} \quad (3.29)$$

$$\leq (M - c\alpha) \frac{u-v}{d^2} \leq 0$$

(here θ do notes the intermediate point in the segment). We need a second result which is the following.

Lemma 3.22 *Let $\alpha \geq 3$, then there exists a unique solution $u \in W_0^{1,p}(\Omega)$, to the linear problem*

$$\begin{cases} -\Delta u + M \frac{u}{d(x)^2} &= h \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.30)$$

for $p < \frac{1+\alpha}{\alpha-1}$. where $h : \Omega \rightarrow \mathbb{R}_+$ is continuous function and satisfies $h(x) \leq cd(x)^{\frac{-2\alpha}{1+\alpha}}$, and

$$\|u\|_{W_0^{1,p}(\Omega)} \leq c \|h\|_{W^{-1,p}(\Omega)}.$$

Proof. (see the theorem 2.6 with $\beta = \frac{2\alpha}{1+\alpha} < 2$), then there exist a unique solution $u \in W_0^{1,p}(\Omega)$ where $p < \frac{\alpha+1}{\alpha-1}$.

Proof of Theorem 3.20 Problem (P') is obviously equivalent to the nonlinear problem

$$\begin{cases} -\Delta u + M \frac{u}{d(x)^2} &= \frac{1}{u^\alpha} + M \frac{u}{d(x)^2} \text{ in } \Omega \\ u &= 0 \text{ on } \partial\Omega \end{cases}$$

By the lemma 3.22 and lemma 3.21 we construct the following iterative scheme starting from the subsolution u_0 such that $u_{k+1} \in H_0^1(\Omega)$ to be unique weak solution to the linear boundary value problem.

$$\begin{cases} \Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} &= \frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^2} \\ u_{k+1} &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.31)$$

We have to prove that

$$u_0 \leq u_1 \leq \dots u_k \leq \dots \leq u^k \leq \dots \leq u^1 \leq u^0.$$

we have by the definition of subsolution u_0 ,

$$\int_{\Omega} \nabla u_0 \nabla v dx - \int_{\Omega} (u_0^{-\alpha}) dx \leq 0, \quad \forall v \geq 0. \quad (3.32)$$

First we prove that $u_0 \leq u_1$, therefor we substituting k by 0 in (3.31), we have

$$\begin{cases} \Delta u_1 + M \frac{u_1}{d(x)^2} &= \frac{1}{u_0^\alpha} + M \frac{u_0}{d(x)^2} \text{ in } \Omega \\ u_1 &= 0 \text{ on } \partial\Omega \end{cases} \quad (3.33)$$

Subtract (3.33) from (3.32), multiplying by $v = (u_0 - u_1)^+ \in W_0^{1,p'}(\Omega)$, $v \geq 0$ and using integration by parts, we obtain

$$\int_{\Omega} \nabla(u_0 - u_1)^{+2} dx + \int_{\Omega} M \frac{u_1}{d(x)^2} (u_0 - u_1)^{+2} dx \leq 0 \quad (3.34)$$

so $\int_{\Omega} \nabla(u_0 - u_1)^{+2} dx = 0$ which implies $(u_0 - u_1)^+ = 0$ this gives $u_0 \leq u_1$.

Now we have to prove that $u_k \leq u_{k+1}$, for this we assume that

$$u_{k-1} \leq u_k. \quad (3.35)$$

From (3.31) for each $v \in W_0^{1,p'}(\Omega)$, we find

$$\int_{\Omega} \nabla u_{k+1} \nabla v + M \frac{u_{k+1}}{d(x)^2} v dx = \int_{\Omega} \left(\frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^2} \right) v dx$$

and

$$\int_{\Omega} \nabla u_k \nabla v + M \frac{u_k}{d(x)^2} v dx = \int_{\Omega} \left(\frac{1}{u_{k-1}^\alpha} + M \frac{u_{k-1}}{d(x)^2} \right) v dx$$

Subtract the preceding formulas, substituting v by $(u_k - u_{k-1})^+$, we obtain

$$\int_{\Omega} (\nabla(u_k - u_{k-1})^{+2} + \int_{\Omega} \frac{M}{d(x)^2} (u_k - u_{k-1})^{+2}) dx = \int_{\Omega} (G(u_{k-1}) - G(u_k)) (u_k - u_{k-1})^+ dx$$

by lemma 3.21 and 3.35

$$G(u_{k-1}) - G(u_k) \leq 0 \implies u_k \leq u_{k+1}.$$

Next we assume that $u_k \leq u^0$ and prove that $u_{k+1} \leq u^0$, we have

$$-\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} = \frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^2}$$

and

$$-\Delta u^0 + M \frac{u^0}{d(x)^2} \geq \frac{1}{(u^0)^\alpha} + M \frac{u^0}{d(x)^2},$$

subtract the preceding formal, multiplying by $v = (u_{k+1} - u^0)^+ \in W_0^{1,p'}(\Omega)$, using integration by parts, and by lemma 3.21, we obtain

$$\begin{aligned} \int_{\Omega} \left(\nabla(u_{k+1} - u^0)^{+2} + \frac{M}{d(x)^2} (u_{k+1} - u^0)^{+2} \right) dx &\leq \int_{\Omega} (G(u_k) - G(u^0)) (u_{k+1} - u^0)^+ dx \\ &\leq 0. \end{aligned}$$

So $u_{k+1} \leq u^0$. Thus we have established the increasing sequence;

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq u^0.$$

By the same way we establish the second decreasing sequence;

$$u^0 \geq u^1 \geq \dots \geq u^k \geq \dots \geq u_0.$$

It remains only to prove that $u_k \leq u^k$, we have $u_0 \leq u^0$, now assume that $u_{k-1} \leq u^{k-1}$ and prove that $u_{k+1} \leq u^{k+1}$, we have

$$-\Delta u_{k+1} + M \frac{u_{k+1}}{d(x)^2} = \frac{1}{u_k^\alpha} + M \frac{u_k}{d(x)^2}$$

and

$$-\Delta u^{k+1} + M \frac{u^{k+1}}{d(x)^2} \geq \frac{1}{u^{(k)\alpha}} + M \frac{u^k}{d(x)^2}$$

Subtract the preceding formal multiplying by $v = (u_{k+1} - u^{k+1})^+ \in W_0^{1,p'}(\Omega)$, using integration by parts with last hypothesis and lemma 3.21, we find

$$\int_{\Omega} (\nabla(u_{k+1} - u^{k+1})^{+2} + \frac{M}{d(x)^2} (u_{k+1} - u^{k+1})^{+2}) dx \leq 0$$

giving again the result. From this we obtain two monotones sequences satisfying;

$$u_0 \leq u_1 \leq \dots \leq u_k \leq \dots \leq u^k \dots \leq u^1 \leq u^0.$$

Now by the regularity of the problem in lemma 3.22 and the estimates in lemma 3.21, we have

$$\|u_k\|_{W_0^{1,p}(\Omega)} \leq c \left\| \frac{1}{u_{k-1}^\alpha} + \frac{Mu_{k-1}}{d^2} \right\|_{W^{-1,p}(\Omega)} \leq c.$$

Where the constant c is independent of k . Thus there exists again subsequences still do notes u_k such that $u_k \rightharpoonup \underline{u}$ weakly in $W_0^{1,p}(\Omega)$ and strongly in $L^p(\Omega)$, therefor by the inverse Lebesgue theorem the subsequence u_k converges to $\underline{u}$ a.e. in Ω . Next we should pass to the limit in equation (3.31) the weak formulation is

$$\int_{\Omega} (\nabla u_k \nabla v + M \frac{u_k v}{d(x)^2}) dx = \int_{\Omega} (\frac{v}{u_{k-1}^\alpha} + M \frac{u_{k-1} v}{d(x)^2}) dx$$

for any $v \in W_0^{1,p'}(\Omega)$ therefor $\int_{\Omega} \nabla u_{k+1} \nabla v dx \longrightarrow \int_{\Omega} \nabla \underline{u} \nabla v dx$, for the first term in right-hand side by the definition of K we obtain

$$\frac{v}{u_{k-1}^{\alpha}} = d \frac{v}{d(x) u_{k-1}^{\alpha}} \leq d^{1-\frac{2\alpha}{1+\alpha}} \frac{v}{d}.$$

By Holder and Hardy's inequality

$$\int_{\Omega} d^{\frac{1-\alpha}{1+\alpha}} \frac{v}{d} dx \leq \left(\int_{\Omega} \left| \frac{v}{d} \right|^{p'} dx \right)^{\frac{1}{p'}} \left(\int_{\Omega} \left| d^{\frac{1-\alpha}{1+\alpha}} \right|^p dx \right)^{\frac{1}{p}} \leq c \int_{\Omega} |\nabla v|^{p'} dx < \infty$$

and we have $u_k \longrightarrow \underline{u}$ a.e.in Ω . So we can apply the dominate convergence theorem this gives that

$$\int_{\Omega} \frac{1}{u_k^{\alpha}} v \longrightarrow \int_{\Omega} \frac{1}{\underline{u}^{\alpha}} v dx, \quad k \rightarrow \infty.$$

For the second terms on both sides we have to prove that

$$\int_{\Omega} M \frac{u_{k-1} v}{d^2} dx \longrightarrow \int_{\Omega} M \frac{\underline{u} v}{d^2} dx, \quad k \rightarrow \infty$$

We have

$$M \frac{u_{k-1} v}{d^2} = M \frac{v}{d} \frac{u_{k-1}}{d}$$

By the definition of K we obtain

$$\left| \frac{u_{k-1}}{d} \right| \leq c d^{\frac{1-\alpha}{1+\alpha}}$$

By Holder and Hady's inequality

$$c \int_{\Omega} \frac{v}{d} d^{\frac{1-\alpha}{1+\alpha}} dx \leq c \left(\int_{\Omega} |\nabla v|^{p'} dx \right)^{\frac{1}{p'}} \left(\int_{\Omega} \left| d^{\frac{1-\alpha}{1+\alpha}} \right|^p dx \right)^{\frac{1}{p}} < \infty.$$

Using as above the dominate convergence theorem is giving again the result.

3.7.1 Ordered of Sub and Supersolutions

Theorem 3.23 *There exist a subsolution u_0 (resp a supersolution u^0) to the problem (P') satisfying the conditions in theorem (3.20).*

Proof. It is the same with the preceding section. So we can easily conclude the theorem (3.20). which gives the theorem (3.19) clearly.

3.8 Uniqueness for $\alpha \geq 3$

Theorem 3.24 *The problem (P') has a unique solutions $u \in W_0^{1,p}(\Omega)$.*

Proof. Let $u, v \in W_0^{1,p}(\Omega)$ are two solutions to the problem (P') . First, we assume that $u \leq v$ in Ω . Multiplying (P') for u by $v \in W_0^{1,p'}(\Omega)$ and by $u \in W_0^{1,p'}(\Omega)$ for v , integrating by parts on Ω , we obtain

$$\int_{\Omega} \nabla u \nabla v - \nabla v \nabla u \, dx = \int_{\Omega} \frac{v}{u^{\alpha}} \, dx - \int_{\Omega} \frac{u}{v^{\alpha}} \, dx = 0.$$

Then

$$\int_{\Omega} \frac{v}{u^{\alpha}} \, dx - \int_{\Omega} \frac{u}{v^{\alpha}} \, dx = \int_{\Omega} \frac{v^{\alpha+1} - u^{\alpha+1}}{v^{\alpha} u^{\alpha}} \, dx = 0$$

Since $v \geq u$, $u \equiv v$. Notice that all the above integrals are meaningful.

Indeed, since $u, v \in k$, by Holder's and Hardy's inequalitys we have

$$\int_{\Omega} \frac{v}{u^{\alpha}} dx = \int_{\Omega} \frac{v}{d} \frac{d}{u^{\alpha}} dx \leq \|v\|_{W_0^{1,p'}(\Omega)} c \int_{\Omega} d(x)^{\frac{p(1-\alpha)}{1+\alpha}} dx < \infty.$$

Such that $\frac{p(1-\alpha)}{1+\alpha} > -1$. If now $u \not\leq v$ and $v \not\leq u$, we have $u_0 \leq u, v \leq u^0$.

Then $\underline{u} \leq u, \underline{u} \leq v$ and it follows from above that $\underline{u} = u = v$.

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